Unit 3
GAME THEORY

Lesson 28

Learning Objective:
• **To learn to apply graphical method in game theory.**
• **Generate solutions in functional areas of business and management using graphs.**

Hello students,

Objective:

In this lecture you are going to study
How to solve the zero sum games which do not possess a saddle point using GRAPHICAL SOLUTION ?

Solution of 2 x n and m x 2 Games

Now,
consider the solution of games where either of the players has only two strategies available:

When the player A, for example, has only 2 strategies to choose from and the player B has n, the game shall be of the order 2 x n, whereas in case B has only two strategies available to him and A has m strategies, the game shall be a m x 2 game.

The problem may originally be a 2 x n or a m x 2 game or a problem might have been reduced to such size after applying the dominance rule. In either case, we can use graphical method to solve the problem. By using the graphical approach, it is aimed to reduce a game to the order of 2 x 2 by identifying and eliminating the dominated strategies, and then solve it by the analytical method used for solving such games. The resultant solution is also the solution to the original problem. Although the game value and the optimal strategy can be read off from the graph, we generally adopt the analytical method (for 2 x 2 games) to get the answer.

We shall illustrate the solution of the 2 x n and m x 2 games in turn with the help of the following examples.

Example 1

Solution of a game using graphical approach:

Payoff matrix

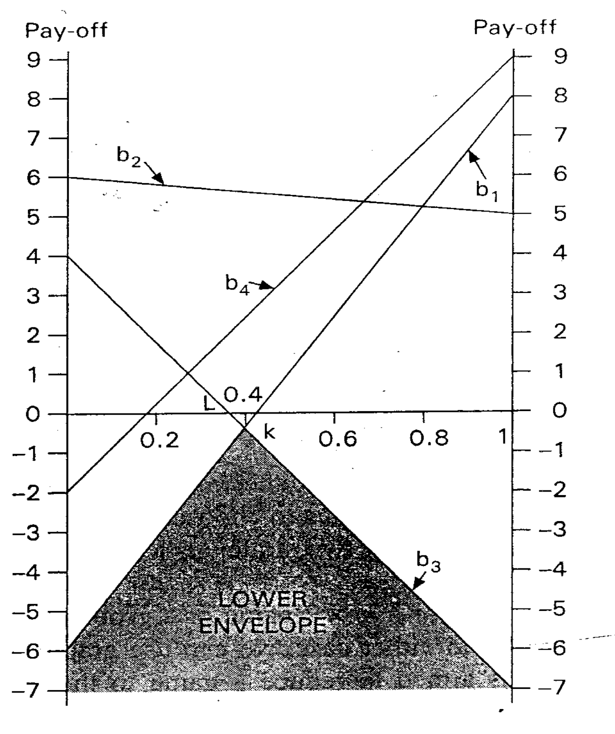
Player A's Strategies	Player B's Strategies			
	B_1	**B_2**	**B_3**	**B_4**
A_1	**8**	**5**	**-7**	**9**
A_2	**-6**	**6**	**4**	**-2**

Here A has two strategies A1 and A2, which suppose, he plays with probabilities p_1 and $1 - p_1$ respectively.

When B chooses to play B1, the expected payoff for A shall be

$$8\, p_1 + (-6)\,(1 - p_1) \text{ or } 14\, p_1 - 6.$$

Similarly, the expected pay-off functions in respect of B2, B3 and B4 can be derived as being $6 - p_1$; $4 - 11\, p_1$; and $11\, p_1 - 2$, "respectively. We can represent these by graphically plotting each pay-off as a function of p_1.



The lines are marked B1, B2, B3 and B4 and they represent the respective strategies. For each value of p_1, the height of the lines at that point denotes the pay-offs of each of B's strategies against (p_1, 1 - p_1) for A. A is concerned with his least pay-off when he plays a particular strategy, which is represented by the lowest of the four lines at that point, and wishes to choose p_1 so as to maximize this minimum pay-off. This is at K in the figure where the lower envelope (represented by the shaded region), the lowest of the lines at point, is the highest. This point lies at the intersection of the lines representing strategies B1 and B3. The distance KL = -0.4 (or –2/5) in the figure represents the game value, V and p_1 = OL (= 0.4 or 2/5) is the optimal strategy for A.

Alternatively, the game can be written as a 2 x 2 game as follows, with strategies A1 and A2 for A, and B1 and B3 for B.

$$
\begin{array}{c} \\ \mathbf{A_1} \\ \mathbf{A_2} \end{array}
\begin{array}{c} \begin{array}{cc} \mathbf{B_1} & \mathbf{B_2} \end{array} \\ \left\| \begin{array}{cc} \mathbf{8} & \mathbf{-7} \\ \mathbf{-6} & \mathbf{4} \end{array} \right\| \end{array}
$$

Here,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-(-6)}{8+4-(-7-6)} = \frac{2}{5}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-(-7)}{8+4-(-7-6)} = \frac{11}{25}$$

The expected value of the game is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{8x4-(-7)(-6)}{8+4-(-7-6)} = \frac{-2}{5}$$

Hence the optimum strategies for the two players are:

$$S_A = \begin{bmatrix} A1 & A2 \\ 2/5 & 3/5 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B1 & B2 & B3 & B4 \\ 11/25 & 0 & 14/25 & 0 \end{bmatrix}$$

Example 2
Solve the following game using the graphical method:

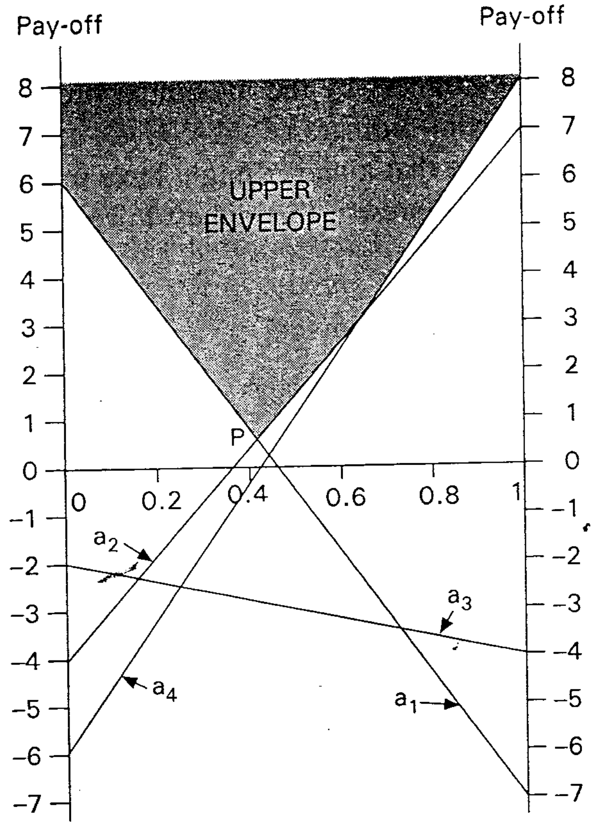
Payoff matrix

Player A's Strategies	Player B's Strategies	
	B_1	**B_2**
A_1	**-7**	**6**
A_2	**7**	**-4**
A3	**-4**	**-2**
A4	**8**	**-6**

Let B play the strategies B1 and B2 with respective probabilities q_1 and $1- q_1$, the expected pay-off for which, when A chooses to play A1, shall be $-7 q_1 + 6(1- q_1)$ or $-13 q_1 + 6$. Similarly, pay-offs in respect of other strategy plays can be determined. These are presented graphically as:

Here we are concerned with the upper envelope, which is formed by the lines representing strategies A1, A2 and A4. The lowest point, P, determines the value of the game.

We obtain the optimal strategies and the value of the game as:



Since the point P is determined by the lines representing strategies A1 and A2, these strategies would enter the solution with non-zero probabilities. Now considering the strategies A1 and A2 for A, and B1 and B2 for B, we have the pay-off matrix given by:

$$
\begin{array}{c c}
 & \begin{array}{cc} \mathbf{B_1} & \mathbf{B_2} \end{array} \\
\begin{array}{c} \mathbf{A_1} \\ \mathbf{A_2} \end{array} & \left\| \begin{array}{cc} \mathbf{-7} & \mathbf{6} \\ \mathbf{7} & \mathbf{-4} \end{array} \right\|
\end{array}
$$

Now,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{-4-7}{-7-4-(6+7)} = \frac{11}{24}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{-4-6}{-7-4-(6+7)} = \frac{5}{12}$$

The expected value of the game is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{(-7)(-4)-(7)(6)}{-7-4-(6+7)} = \frac{7}{12}$$

Hence the optimum strategies for the two players are:

$$S_A = \begin{bmatrix} A1 & A2 & A3 & A4 \\ 11/24 & 13/24 & 0 & 0 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B1 & B2 \\ 5/12 & 7/12 \end{bmatrix}$$

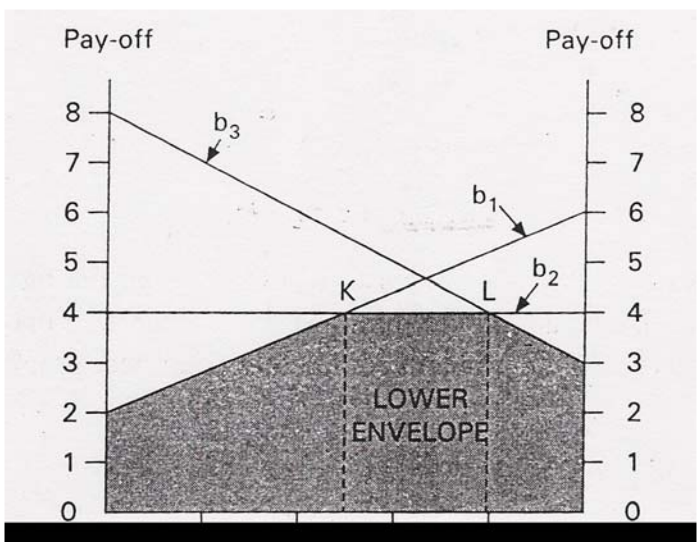
Example 3
Solve the game with the following pay-off:

Payoff matrix

Player A's Strategies	Player B's Strategies		
	B_1	B_2	B_3
A_1	6	4	3
A_2	2	4	8

Solution:

The lower envelope representing the feasible region is shaded. There is no single point representing the highest point. The value of the game is clearly 4 and any value of p_1, between the points represented by K and L shall be optimal for A.



To obtain K,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-2}{6+4-(2+4)} = \frac{2}{4} = 0.5$$

To obtain L,

$$p_1 = \frac{8-4}{4+8-(3+4)} = \frac{4}{5} = 0.8$$

Thus an optimal strategy for A is any pair of (p_1, 1- p_1) where $0.5 \le p_1 \le 0.8$

Now,
Apply this graphical method on some Unsolved Problems yourself

Problem 1. Determine the optimal minimax strategies for each player in the following game.

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{cccc} B1 & B2 & B3 & B4 \end{array} \\ \left\Vert \begin{array}{cccc} -5 & 2 & 0 & 7 \\ 5 & 1 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{array} \right\Vert \end{array}$$

Problem 2. Solve the following games by using maximin - minimax principle whose payoff matrix are given below: Include in your answer:

(i) Strategy selection for each player.

(ii) The value of the game to each player.

(iii) Does the game have a saddle point?

(a) Player A, Player B

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{cccc} B1 & B2 & B3 & B4 \end{array} \\ \left\Vert \begin{array}{cccc} 1 & 7 & 3 & 4 \\ 5 & 6 & 2 & 5 \\ 7 & 4 & 0 & 3 \end{array} \right\Vert \end{array}$$

(b) Player A, Player B

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{cccc} B1 & B2 & B3 & B4 \end{array} \\ \left\Vert \begin{array}{cccc} 3 & -5 & 0 & 6 \\ -4 & -2 & 1 & 2 \\ 5 & -4 & 2 & 3 \end{array} \right\Vert \end{array}$$

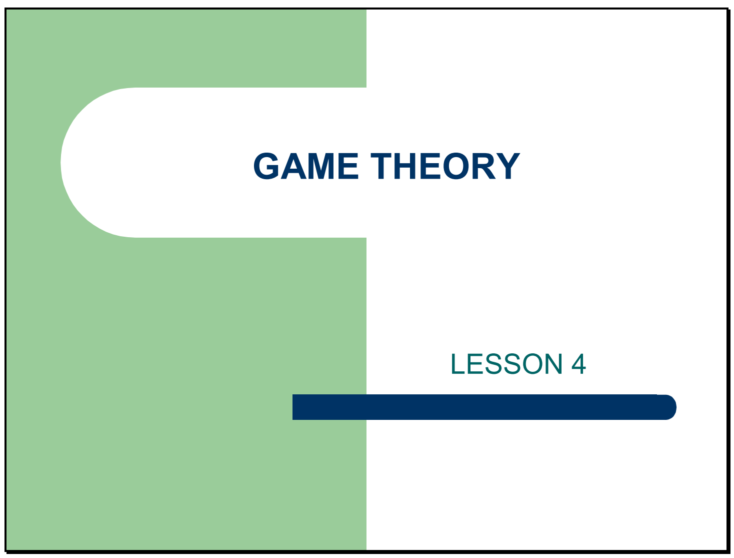
So, now let us summarize today's discussion:

Summary

We have discussed about:

- Graphical solutions
- Solution of 2 x n and m x 2 Games using graphs

Slide 1



Slide 2

GRAPHICAL SOLUTION OF 2 X n AND m X 2 GAMES

- Graphical approach is aimed to reduce a game payoff matrix to the order of 2 x 2 by identifying and eliminating the dominated strategies.