Unit 3
GAME THEORY

Lesson 26

Hello students,

In previous lecture you learned to solve the zero-sum games having saddle point.

Learning Objective:

- **In this lecture you are going to study How to solve the zero sum games which do not possess a saddle point?**

TWO PERSON ZERO SUM GAME (WITHOUT A SADDLE POINT)

You have observed that in game situations which have a saddle point(s) are provided with an adequate theory of how best to play the game. It is possible that there is no saddle point of a game and then it is not possible to find its solution in terms of the pure strategies – the maximin and the minimax.

Games without saddle point are not strictly determined. The solution to such problems calls for employing mixed strategies i.e. to solve such games both the players must determine an optimal mixture of strategies to find a saddle point. A mixed strategy therefore represents a combination of two or more strategies that are selected one at a time, according to pre-determined probabilities. Thus in employing a mixed strategy, a player decides to mix his choices among several alternatives in a certain ratio.

Consider this with the help of an example

EXAMPLE 1

Determine the optimal strategies for the players and value of the game from the following payoff matrix.

		Player B	
Player A			
		B_1	B_2
A_1		8	-7
A_2		-6	4

Solution:
The given problem does not have a saddle point. Therefore, the method of saddle point is not sufficient to determine optimal strategies. If A plays A_1 then B would play B_2 while if A plays A_2 then B would choose B_1 to play. So if B knows what choice A will make then B can ensure that he gains by choosing a strategy, opposite to the one desired by A. Thus it is important for A to make it difficult for B to guess as to what choice he is going to make. Similarly, B would like to make it very difficult for A to assess the strategy he is likely to adopt.

Now suppose that A plays strategy A_1, with probability p_1 and plays strategy A_2 with probability $p_2 = 1 - p_1$. If B plays strategy B_1, then A's expected payoff can be determines by the figures in the first column of the payoff matrix as:

Expected payoff (if B plays B_1) $= 8p_1 - 6p_2$
$= 8p_1 - 6(1-p_1)$

Similarly, If B plays B_2, the expected payoff to A can be determined as:

Expected pay off (if B plays B_2) $= -7p_1 + 4p_2$
$= -7p_1 + 4(1-p_1)$

Now, we determine a value of p_1 so that the expected pay – off for A is the same irrespective of the strategy adopted by B

Thus, $8p_1 - 6(1-p_1) = -7p_1 + (1-p_1)$

$\Rightarrow 8p_1 - 6 + 6p_1 = -7p_1 + 4 - 4p_1$

$$\Rightarrow \quad p_1 = \frac{10}{25} = \frac{2}{5}$$

$$p_2 = 1 - p_1 = \frac{3}{5}$$

A would do best to adopt the strategies A_1 and A_2, choosing in random manner, in the ratio 2:3 (2/5 and 3/5) .

The expected pay off for A using this mixed strategy is given by.

$$8 \times \frac{2}{5} - 6 \times \frac{3}{5} = -\frac{2}{5}$$

or

$$-7 \times \frac{2}{5} - 4 \times \frac{3}{5} = \frac{2}{5}$$

Thus he will have a loss of 2/5 per play.

Now, we determine the mixed strategy for B in a similar manner. Let us suppose B plays strategy B_1, with probability q_1 and strategy B_2 with probability $q_2 = 1-q_1$
Then,

Expected payoff (given that A plays A_1) $= 8q_1 - 7q_2$
$= 8q_1 - 7(1-q_1)$

Expected payoff (given that A plays A_2) $= -6q_1 + 4q_2$
$= -6q_1 + 4(1-q_1)$

The value of q_1 so that the expected payoff for B is same irrespective of the strategy of A is obtained as.

$$8q_1 - 7(1-q_1) = -6q_1 + 4(1-q_1)$$

$$\Rightarrow \quad 8q_1 - 7 + 7q_1 = -6q_1 + 4 - 4q_1$$

$$\Rightarrow \quad q_1 = \frac{11}{25}, \quad q_2 = \frac{14}{25}$$

Thus B should play strategies B_1 and B_2 in the ratio 11: 14 in a random manner.

B's expected payoff per play shall be:

$$8 \times \frac{11}{25} - 7 \times \frac{14}{25} = \frac{-10}{25} = \frac{-2}{5}$$

$$-6 \times \frac{11}{25} + 4 \times \frac{14}{25} = \frac{-10}{25} = \frac{-2}{5}$$

i.e. B shall gain 2/5 per play

We conclude that A and B should both use mixed strategies given as

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{11}{25} & \frac{14}{25} \end{bmatrix}$$

and value of the game = $\frac{-2}{5}$

IN GENERAL, for solving a 2 x 2 game without a saddle point, in which each of the players, say A and B have strategies A_1 and A_2 and B_1 and B_2 respectively. If A chooses strategy A_1 with the probability p_1 and A_2 with probability $p_2 = 1-p_1$ and B plays strategy B_1 with probability q_1 and strategy B_2 with probability $q_2 = 1-q_1$

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

And their payoff matrix is given as

A's Strategies	B's strategies B1	B2
A1	a11	a12
A2	a21	a22

Then the following formulas are used to find the value of the game and optimal strategies:

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} \quad ; p_2 = 1-p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} \quad ; q_2 = 1-q_1$$

and

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})}$$

EXAMPLE 2

Solve the following game and determine the value of the game:

		Player Y Strategy-1	Strategy-2
Player X	Strategy-1	4	1
	Strategy-2	2	3

Solution:

Clearly, the pay-off matrix does not possess any saddle point. The two players, therefore, use mixed strategies. Let

p_1 = probability that player X uses strategy 1.

q_1 = probability that player Y uses strategy 1.

Then $1 - p_1$ = probability that player X uses strategy 2

$1 - q_1$ = probability that player Y uses strategy 2

If player Y selects strategy 1 and player X selects the options with probabilities p_1 and $1 - p_1$, then expected pay-off to player X would be

= 4 (probability of player X selecting strategy 1) + 2 (probability of player X selecting strategy 2)

$= 4\, p_1 + 2\,(1 - p_1) = 2\, p_1 + 2$

If player Y selects strategy 2 then expected pay-off to player X will be

$=1\, p_1 + 3(1 - p_1) = -2\, p_1 + 3$

The probability p_1 should be such that expected pay-offs under both conditions are equal, i.e., $2\, p_1 + 2 = -2\, p_1 + 3$ or $p_1 = 1/4$

i.e., player X selects strategy 1 with a probability of 1/4 or 25% of the time and strategy 2, 75% of the time.

Similarly expected pay-offs from player Y can be computed as follows:

Expected pay-offs from player Y when player X selects strategy 1 = Expected pay-off from player Y when player X selects strategy 2.

or $4\, q_1 + 1(1 - q_1) = 2\, q_1 + 3(1 - q_1)$

or $q_1 =1/2$ and $1 - q_1 = 1/2$

This implies that player Y selects each strategy with equal probability, Le., 50% of the time he chooses strategy 1 and 50% of the time strategy 2.

Value of game = (Expected profits to player X when player Y uses strategy 1) x Prob. (player Y using strategy 1) + (Expected profits to player X when player Y uses strategy 2) x Prob. (player Y using strategy 2).

$= \{4x\, p_1 + 2(1 - p_1)\}q_1 + \{1x\, p_1 + 3(1 - p_1)\}(1 - q_1)$

$=\{4x1/4 + 2(1-1/4)\}1/2 + \{1x1/4 + 3(1-1/4)\}(1-1/2)$

$= 10/4$

Alternatively: The optimum mixed strategies for player X and Y are determined by :

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{3-2}{4+3-(2+1)} = \frac{1}{4}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-2}{4+3-(2+1)} = \frac{1}{2}$$

The expected value of the game is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4x3-2x1}{4+3-(2+1)} = \frac{10}{4}$$

Hence the optimum strategies for the two players are:

$$S_X = \begin{bmatrix} X1 & X2 \\ 1/4 & 3/4 \end{bmatrix} \qquad S_Y = \begin{bmatrix} Y1 & Y2 \\ 1/2 & 1/2 \end{bmatrix}$$

EXAMPLE 3

Determine the optimal strategies for the players and value of the game from the following payoff matrix.

Player B

Player A

	B_1	B_2
A_1	5	1
A_2	3	4

Solution:

The given problem does not have a saddle point. Therefore, the method of saddle point is not sufficient to determine optimal strategies.

Let the payoff matrix is given by

A's Strategies	B's strategies B1	B2
A1	a11	a12
A2	a21	a22

Then the optimal mixed strategies are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-3}{5+4-(1+3)} = \frac{1}{5}$$

$$p_2 = 1-p_1 = 4/5$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-1}{5+4-(1+3)} = \frac{3}{5}$$

$$q_2 = 1-q_1 = 2/5$$

The expected value of the game is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{(5x4)-(1x3)}{5+4-(1+3)} = \frac{17}{5}$$

Hence the optimal mixed strategies are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{bmatrix}$$

EXAMPLE 4

Determine the optimal strategies for the players and value of the game from the following payoff matrix.

Player B

Player A

	B_1	B_2
A_1	4	-4
A_2	-4	4

Solution:

The given problem does not have a saddle point. Therefore, the method of saddle point is not sufficient to determine optimal strategies.

Let the payoff matrix is given by

B's strategies

A's Strategies

	B1	B2
A1	a11	a12
A2	a21	a22

Then the optimal mixed strategies are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-(-4)}{4+4-(-4-4)} = \frac{1}{2}$$

$$p_2 = 1-p_1 = 1/2$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{4-(-4)}{4+4-(-4-4)} = \frac{1}{2}$$

$$q_2 = 1-q_1 = 1/2$$

The expected value of the game is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} = \frac{(4x4)-(-4x(-4))}{4+4-(-4+(-4))} = 0$$

Hence the optimal mixed strategies are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ 1/2 & 1/2 \end{bmatrix}$$

Now,
Apply this method on an Unsolved Problem yourself

Q.1. Determine the strategies for player. A and B in the following game. Also indicate the value of the game.

	B_1	B_2
A_1	20	25
A_2	30	-15

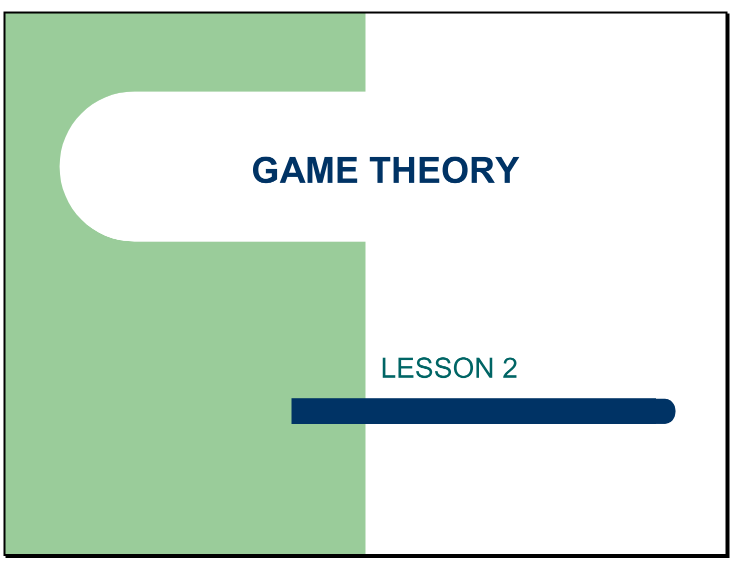
So, now let us summarize today's discussion:

Summary

We have discussed about:

- Factors influencing game theory models.
- Identification of two person zero sum games.
- Minimax and Maximin Principle
- Importance and application of saddle point.

Slide 1

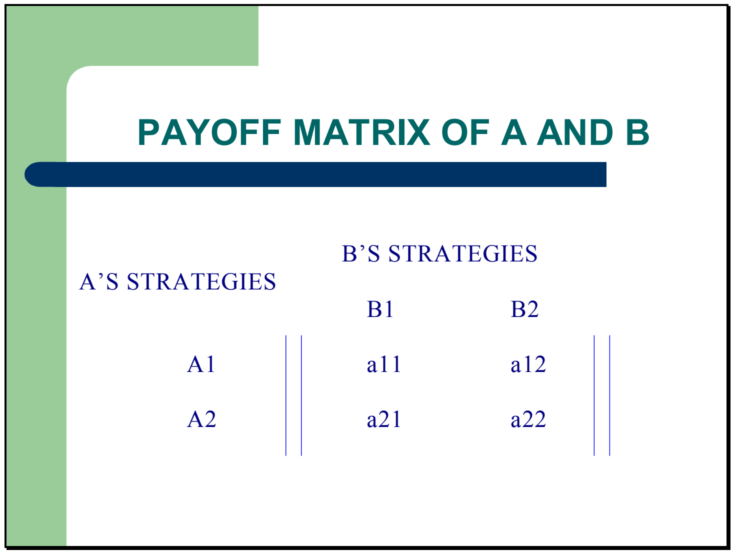


Slide 2

TWO PERSON ZERO SUM GAME (WITHOUT A SADDLE POINT)

- Player A has strategy A_1 and A_2
- Player B has strategy B_1 and B_2
- A chooses strategy A_1 with the probability p_1 and A_2 with probability $p_2 = 1-p_1$
- B plays strategy B_1 with probability q_1 and strategy B_2 with probability $q_2 = 1-q_1$

Slide 3


PAYOFF MATRIX OF A AND B
B'S STRATEGIES
A'S STRATEGIES
B1 B2
A1 a11 a12
A2 a21 a22

Slide 4

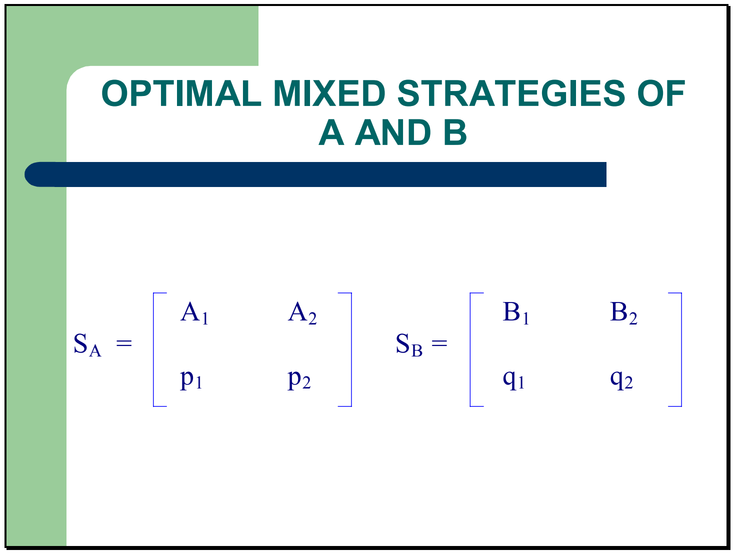
Evaluation of Probabilities

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12} + a_{21})} \quad ; p_2 = 1-p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})} \quad ; q_2 = 1-q_1$$

and $$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12} + a_{21})}$$

Slide 5


OPTIMAL MIXED STRATEGIES OF A AND B
$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$
$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$