Unit 3
GAME THEORY

Lesson 25

Learning Objective:

This chapter aims at showing you

- How games can be analyzed;
- How games can be related to numbers
- To demonstrate the theory of a representative collection of [mathematical] games
- To show how new games can be investigated and related to other games.

Hello students,

In this course I will give an overview into methods for `solving games' as well as give examples for how games are used to model various scenarios.

Let us begin with the introduction of game theory

What is game theory?

Game theory is the study of how optimal strategies are formulated in conflict. It is concerned with the requirement of decision making in situations where two or more rational opponents are involved under conditions of competition and conflicting interests in anticipation of certain outcomes over a period of time.

You are surely aware of the fact that

In a competitive environment the strategies taken by the opponent organizations or individuals can dramatically affect the outcome of a particular decision by an organization.

In the automobile industry, for example,
the strategies of competitors to introduce certain models with certain features can dramatically affect the profitability of other carmakers.

So in order to make important decisions in business, it is necessary to consider what other organizations or individuals are doing or might do. Game theory is a way to consider the impact of the strategies of one, on the strategies and outcomes of the other.

In this you will determine the rules of rational behavior in the game situations, in which the outcomes are dependent on the actions of the interdependent players.

A GAME refers to a situation in which two or more players are competing. It involves the players (decision makers) who have different goals or objectives. They are in a situation in which there may be a number of possible outcomes with different values to them. Although they might have some control that would influence the outcome, they do not have complete control over others. Unions striking against the company management, players in a chess game, firm striving for larger share of market are a few illustrations that can be viewed as games.

Now I throw some light on the evolution of game theory

AN OUTLINE OF THE HISTORY OF GAME THEORY

Some game-theoretic ideas can be traced to the 18th century, but the major development of the theory began in the 1920s with the work of the mathematician Emile Borel (1871Œ1956) and the polymath John von Neumann (1903Œ57). A decisive event in the development of the theory was the publication in 1944 of the book Theory of games and economic behavior by von Neumann and Oskar Morgenstern, which established the foundations of the field. In the early 1950s, John F. Nash developed a key concept (Nash equilibrium) and initiated the game-theoretic study of bargaining.

Soon after Nash's work, game-theoretic models began to be used in economic theory and political science, and psychologists began studying how human subjects behave in experimental games. In the 1970s game theory was first used as a tool in evolutionary biology. Subsequently, game-theoretic methods have come to dominate microeconomic theory and are used also in many other fields of economics and a wide range of other social and behavioral sciences. The 1994 Nobel Prize in economics was awarded to the game theorists John C. Harsanyi (1920Œ2000), John F. Nash (1928Œ), and Reinhard Selten (1930Œ).

GAME THEORY MODELS

You will gradually learn that the models in the theory of games can be classified depending upon the following factors:

Number of players

It is the number of competitive decision makers, involved in the game. A game involving two players is referred to as a "Two-person game". However if the number of players is more (say n>2) then the game is called an n-person game.

Total Payoff

It is the sum of gains and losses from the game that are available to the players. If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that the sum of the gains and losses equals zero then the game is said to be a zero-sum game. There are also games in which the sum of the players' gains and losses does not equal zero, and these games are denoted as non-zero-sum games.

Strategy

In a game situation, each of the players has a set of strategies available. The strategy for a player is the set of alternative courses of action that he will take for every payoff (outcome) that might arise. It is assumed that the players know the rules governing the choices in advance. The different outcomes resulting from the choices are also known to the players in advance and are expressed in terms of the numerical values (e.g. money, market share percentage etc.)

Strategy may be of two types:

(a) Pure strategy

If the players select the same strategy each time, then it is referred as pure – strategy. In this case each player knows exactly what the opponent is going to do and the objective of the players is to maximize gains or to minimize losses.

(b) Mixed Strategy

When the players use a combination of strategies with some fixed probabilities and each player kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is probabilistic situation and objective of the player is to maximize expected gains or to minimize losses strategies. Mixed strategy is a selection among pure strategies with fixed probabilities.

Optimal Strategy

A strategy which when adopted puts the player in the most preferred position, irrespective of the strategy of his competitors is called an optimal strategy. The optimal strategy involves maximal pay-off to the player.

The games can also be classified on the basis of the number of strategies. A game is said to be finite if each player has the option of choosing from only a finite number of strategies otherwise it is called infinite.

Now we move on to the most important part of this chapter

TWO PERSON ZERO SUM GAME

A game which involves only two players, say player A and player B, and where the gains made by one equals the loss incurred by the other is called a two person zero sum game.

For example,

If two chess players agree that at the end of the game the loser would pay Rs 50 to the winner then it would mean that the sum of the gains and losses equals zero. So it is a two person – zero sum game.

All this will require you to know about Payoff matrix of the game:

The payoffs (a quantitative measure of satisfaction which a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies, can be represented in the form of a matrix, called the payoff matrix.

Since the game is zero sum, the gain of one player is equal to the loss of the other and vice-versa.

This means, one players' payoff table would contain same amounts in payoff table of the other player with the opposite sign. So it is sufficient to construct payoff table for any one of the players.

If Player A has m strategies represented as $A_1, A_2,$ -------- , A_m and player B has n strategies represented by $B_1, B_2,$ ------- ,B_n. Then the total number of possible outcomes is m x n. Here it is assumed that each player knows not only his own list of possible courses of action but also those of his opponent. It is assumed that player A is always a gainer whereas player B a loser. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy i and player B chooses strategy j.

Then the payoff matrix is :

Player A's Strategies	**Player B's Strategies**		
	B_1	B_2	B_n
A_1	a_{11}	a_{12}	........a_{1n}
A_2	a_{21}	a_{22}	.... a_{2n}
.	.		.
.	.		.
.	.		.
A_m	a_{m1}	a_{m2}	.... a_{mn}

By convention, the rows of the payoff Matrix denote player A's strategies and the columns denote player B's strategies. Since player A is assumed to be the gainer always so he wishes to gain a payoff a_{ij} as large as possible and B tries to minimize the same.

Now consider a simple game

Suppose that there are two lighting fixture stores, X and Y. The respective market shares have been stable until now, but the situation then changes. The owner of store X has developed two distinct advertising strategies, one using radio spots and the other newspaper advertisements. Upon hearing this, the owner of store Y also proceeds to prepare radio and newspaper advertisements.

Payoff matrix

Player X's Strategies	Player Y's Strategies: Y_1 (Use radio)	Y_2 (Use newspaper)
X_1 (Use radio)	**2**	**7**
X_2 (Use newspaper)	**6**	**−4**

The 2 x 2 payoff matrix shows what will happen to current market shares if both stores begin advertising. The payoffs are shown only for the first game player X as Y's payoffs will just be the negative of each number. For this game, there are only two strategies being used by each player X and Y.

Here a positive number in the payoff matrix means that X wins and Y loses. A negative number means that Y wins and X loses. This game favors competitor X, since all values are positive except one. If the game had favored player Y, the values in the table would have been negative. So the game is biased against Y. However since Y must play the game, he or she will play to minimize total losses.

From this game can you state the outcomes of each player?

GAME OUTCOMES

X's Strategy	Y's Strategy	Outcome (% Change in market share)
X1 (use radio)	Y1 (use radio)	X wins 2 and Y loses 2
X1 (use radio)	Y2 (use newspaper)	X wins 7 and Y loses 7
X2 (use newspaper)	Y1 (use radio)	X wins 6 and Y loses 6
X2 (use newspaper)	Y2 (use newspaper)	X loses 4 and Y wins 4

You must have observed in our discussion that we are working with certain **ASSUMPTIONS OF THE GAME** such as:

1. Each player has to choose from a finite number of possible strategies. The strategies for each player may or may not be the same.
2. Player A always tries to maximize his gains and players B tries to minimize the losses.
3. The decision by both the players is taken individually prior to the play without any communication between them.
4. The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
5. Each player knows not only his own list of possible course of action but also of his opponent.

Now I would like you to tell what principle do we follow in solving a zero sum game

MINIMAX AND MAXIMIN PRINCIPLE

You must be aware that the selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem in playing games. The objective of the study is to know how these players must select their respective strategies so that they could optimize their payoff. Such a decision making criterion is referred to as the minimax – maximin principle.

For player A

minimum value in each row represents the least gain to him if he chooses that particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives maximum gain among the row minimum values. This choice of player A is called the maximin criterion and the corresponding gain is called the maximin value of the game.

Similarly, for player B

who is assumed to be the loser, the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written as column maxima. He will select that strategy which gives minimum loss among the column maximum values. This choice of player B is called the minimax criterion, and the corresponding loss is the minimax value of the game.

If the maximin value equals the minimax value, then the game is said to have a saddle point and the corresponding strategies are called optimal strategies. The amount of payoff at an equilibrium point is known as the VALUE of the game. A game may have more than one saddle point or no saddle point.

To illustrate the minimax – maximin principle,
consider a two person zero – sum game with the given payoff matrix for player A.

Payoff matrix

Player A's Strategies	Player B's Strategies	
	B_1	**B_2**
A_1	**4**	**3**
A_2	**8**	**6**
A_3	**5**	**4**

Let the pure strategies of the two players be denoted by

$$S_A = \{A_1, A_2, A_3\} \text{ and}$$

$$S_B = \{B_1, B_2\}$$

Suppose that player A starts the game knowing that whatever strategy he adopts, B will select that particular counter strategy which will minimize the payoff to A. Thus if A selects the strategy A_1 then B will select B_2 as his corresponds to the minimum payoff to A corresponding to A_1. Similarly, if A chooses the strategy A_2, he may gain 8 or 6 depending upon the choice of B.

If A chooses A_2 then A can guarantee a gain of at least min $\{8,6\} = 6$ irrespective of the choice of B. Obviously A would prefer to maximize his minimum assured gains. In this example the selection of strategy A_2 gives the maximum of the minimum gains to A.

This gain is called as maximin value of the game and the corresponding strategy as maximum strategy.

On the other hand, player B prefers to minimize his losses. If he plays strategy B_1 his loss is at the most max {4,8,5} = 8 regardless of the strategy selected by A. If B plays B2 then he loses no more than max {3,6,4} = 6. 'B' wishes to minimize his maximum possible losses. In this example, the selection of strategy B_2 gives the minimum of the maximum losses of B, this loss is called the minimax value of the game and the corresponding strategy the minimax strategy the minimax strategy.

In this example,

$$\text{Maximin value } (\underline{V}) = \text{minimax value } (\overline{V})$$

i.e. maximum {row minima} = minimum value {column maxima}

or $$\max_{i} \{ r_i \} = 6 = \min \{ C_j \}$$

or $$\max_{I} \{ \min_{j} [a_{ij}] \} = 6 = \min_{j} [\max_{i} \{ a_{ij} \}]$$

I = 1,2,3 and j = 1,2

Let me summarize this process with certain rules:

RULES FOR DETERMINING A SADDLE POINT

STEP 1: Select the minimum element of each row of the payoff matrix and mark them as (*) . This is row minima of the respective row.

STEP 2: Select the greatest element of each column of the payoff matrix and mark them as (°) . This is column maxima of the respective column.

STEP 3: If there appears an element in the payoff matrix marked with (*) and (°) both, then this element represents the value of the game and its position is a saddle point of the payoff matrix.

Note: A game is said o be fair, if maximin value = 0 = minimax value

i.e. $\underline{V} = 0 = \overline{V}$

2. A game is said to be strictly determinable,

if $\underline{V} = V = \overline{V}$

3. In general,
maximin value ($\underline{V}$) $\leq$ V $\geq$ minimax value ($\overline{V}$)

Do you know how to implement these rules now?
Try on the following example

EXAMPLE 1

For the game with payoff matrix

Player B

Player A

$$\begin{Vmatrix} -2 & 3 & -4 \\ 7 & 5 & -5 \end{Vmatrix}$$

determine the best strategies for players A and B. Also determine the value of the game. Is this game (i) fair? (ii) strictly determinable?

Here I give you the exact solution of example 1. Check it with your own solution.
Solution: Applying the rule of finding out the saddle point, we obtain the saddle point which as marked with (*) and (º) both.

	Player B strategies			Row minima
Player A Strategies	-2	3	-4 *º	-4
	7º	5º	-5*	-5
Column maxima	7	5	-4	

The payoffs marked with (*) represent the minimum payoff in each row and those marked with (º) represent the maximum payoff in each column of the payoff matrix. The largest quantity of row minima represents ($\underline{V}$) maximin value and the smallest quantity of column maxima represents ($\overline{V}$) minimax value.

Thus we have,

Maximin value ($\underline{V}$) = - 4 = ($\overline{V}$) minimax value this value is referred to as **saddle point.**

The payoff amount in the saddle point position is also called the value of the game. For this game, value of the game is V=-4, for players A The game is strictly determinable and not fair.

Let us take up some more solved examples

EXAMPLE 2

Find the range of values of p and q that will make the payoff element a_{22} a saddle point for the game whose payoff matrix is:

		Player B		
Player A	2	4	5	
	10	7	q	
	4	p	6	

Solution: First ignoring the values of p and q in the payoff matrix, determine the maximin and minimax values of the payoff matrix.

		Player B		Row minima
Player A	2*	4	5	2
	10º	7*º	q	7
	4*	p	6	4
Column maxima	10	7	6	

Since there does not exist a unique saddle point, the element a_{22} will be a saddle point only when $p \leq 7$ and $q \geq 7$

EXAMPLE 3

A company management and the labour union are negotiating a new 3 year settlement . Each of these has 4 strategies.

I: Hard and aggressive bargaining
II: Reasoning and logical approach
III: Legalistic strategy
IV: Conciliatory approach.

The costs to the company are given for every pair of strategy choice.

		Company Strategies			
		I	II	II	IV
	I	20	15	12	35
Union	II	25	14	8	10
Strategies	III	40	2	10	5
	IV	-5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution: Obtain the saddle point by applying the rules to find a saddle point.

		Company Strategies				Row minima
		I	II	II	IV	
	I	20	15º	12*º	35º	12
Union	II	25	14	8*	10	8
Strategies	III	40º	2*	10	5	2
	IV	-5*	4	11	0	-5
Column maxima		40	15	12	35	

Maximin = Minimax = Value of game = 12

The company incurs costs and hence its strategy is to minimize maximum losses. For the union, negotiation results in gain, hence union strategy aims at maximizing minimum gains.
Since there exists are saddle point, strategies are pure and given as: Company will always adopt strategy III – legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

EXAMPLE 4
What is the optimum strategy in the game described by the matrix.

-5	3	2	10
5	5	4	6
-4	-2	0	-5

Solution: We determine the saddle point

PLAYER B

PLAYER A

	I	II	III	IV
I	-5	3	2*	10º
II	5º	5º	4*º	6
III	-4	-2	0	-5*

Maximin = Minimax = Value of the game = 4

Hence the solution to the game is

1. The optimal strategy for player A is II
2. The optimal strategy for player B is III
3. The value of the game is 5.

EXAMPLE 5

Shruti Ltd has developed a sales forecasting function for its products and the products of its competitor, Purnima Ltd. There are four strategies S1, S2, S3 and S4 available to Shruti Ltd. and three strategies P1, P2 and P3 to Purnima Ltd. The pay-offs corresponding to all the twelve combinations of the strategies are given below. From the table we can see that, for example, if strategy S1 is employed by Shruti Ltd. and strategies P1 by Purnima Ltd., then there shall be a gain of Rs. 30,000 in quarterly sales to the former. Other entries can be similarly interpreted.

Considering this information, state what would be the optimal strategy for Shruti Ltd.? Purnima Ltd.? What is the value of the game? Is the game fair?

Purnima Ltd.'s Strategy

		P1	P2	P3
Shruti's Strategy	S1	30,000	-21,000	1,000
	S2	18,000	14,000	12,000
	S3	-6,000	28,000	4,000
	S4	18,000	6,000	2,000

Solution: For determining the optimal strategies for the players' we shall determine if saddle point exists for this

Purnima's Strategy

		P1	P2	P3	Row Minima
Shruti's Strategy	S1	30,000º	-21,000*	1,000	-21,000
	S2	18,000	14,000	12,000*º	12,000
	S3	- 6,000*	28,000º	4,000	-6,000
	S4	18,000	6,000	2,000*	2,000
Column maxima		30,000	28,000	12,000	

Here saddle point exists at the intersection of S2 and P3. These represent optimal policies, respectively, of Shruti Ltd. and Purnima Ltd.
Correspondingly, the value of game, V=12,000. Since $V \neq 0$, the game is not a fair one.

EXAMPLE 6
Solve the game whose pay off matrix is given by

Player B

		B1	B2	B3
Player A	A1	1	3	1
	A2	0	-4	-3
	A3	1	5	-1

Solution :

		Player B			
		B1	B2	B3	Row minima
	A1	1*°	3	1*°	1
Player A	A2	0	-4*	-3	-4
	A3	1°	5°	-1*	-1
Column maxima		1	5	1	

Maxi (minimum) = Max (1, -4, -1) = 1
Mini (maximum) = Min (1,5,1) = 1
ie., Maximin value $\underline{v}$= 1 = Minimax value $\overline{v}$

Therefore, Saddle point exists. The value of the game is the saddle point which is 1. The optimal strategy is the position of the saddle point and is given by (A1,B1).

EXAMPLE 7

For what value of λ, the game with the following matrix is strictly determinable?

		Player B		
		B1	B2	B3
	A1	λ	6	2
Player A	A2	-1	λ	-7
	A3	-2	4	λ

Solution Ignoring the value of λ, the payoff matrix is given by

		Player B			
		B1	B2	B3	Row minima
	A1	λ	6	2	2
Player A	A2	-1	λ	-7	-7
	A3	-2	4	λ	-2
	Column maxima	-1	6	2	

The game is strictly determinable, if

$$v = \underline{v} = v$$

Hence $\underline{v} = 2$

$v = -1$

$$\Rightarrow -1 \leq \lambda \leq 2$$

Now, I will give you some unsolved problems

Problem 1: Burger Giant and Pizza mania are competing for a larger share of the fast-food market. Both are contemplating the use of promotional coupons. If Burger Giant does not spend any money of promotional coupons, it will not lose any share of the market if Pizza mania also does not spend any money on promotional coupons. Burger Giant will lose 4 percent of the market if Pizza mania spends Rs. 2500 on coupons, and it will lose 6 percent of the market if Pizza mania spends Rs. 3000 on coupons. If Burger Giant spends Rs. 2500 on coupons, it will gain 3 percent of the market if pizza mania spends Rs. 0, and it will gain 2 percent if Pizza mania Spends Rs. 2500, and it will lose 1 percent of the market if Pizza mania spends Rs. 3000. If Burger Giant Spends Rs. 3000, it will gain 5% of the market if Pizza mania spends Rs 0, it will gain 3% of the market if Pizza mania spends Rs. 2500, and gains 2% of the market if Pizza mania spends Rs. 3000.

(a) Develop a payoff table for this game.
(b) Determine the strategies that Burger Giant and Pizza mania should adopt.
(c) What is the value of this game?

Problem 2: The two major scooter companies of India, ABC and XYZ, are competing for a fixed market. If both the manufacturers make major model changes in a year, then their shares of the market do not change. Also, if they both do not make major model changes, their shares of the market remain constant. If ABC makes a major model change and XYZ does not, then ABC is able to take away a% of the market away from XYZ, and if XYZ makes a major model change ABC does not, XYZ is able to take away b% of the market away from ABC. Express it as a two-by-two game and solve for the optimal strategy for each of the producers.

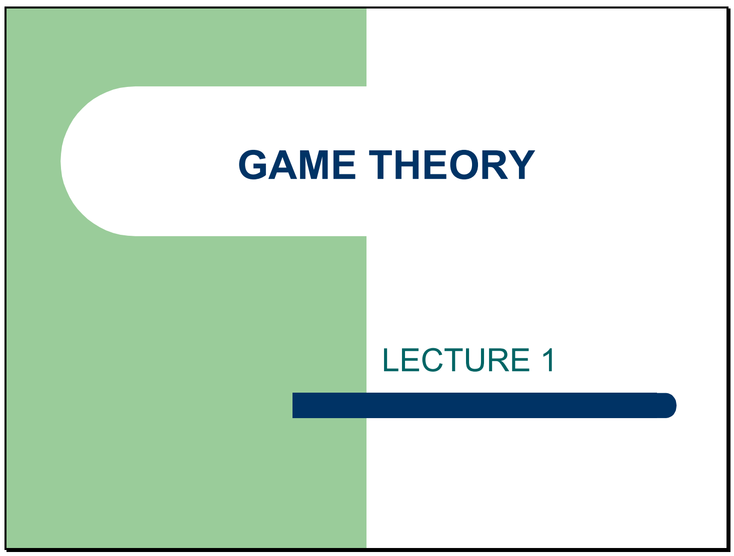
So, now let us summarize today's discussion:

Summary

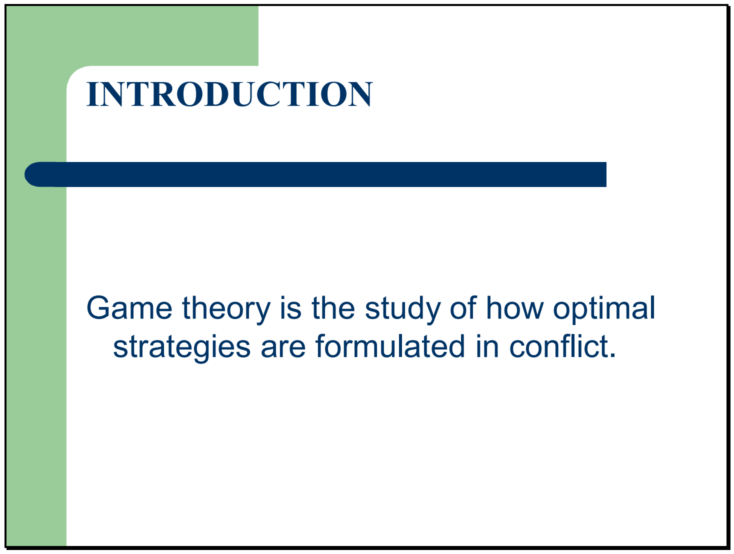
We have discussed about:

- Factors influencing game theory models.
- Identification of two person zero sum games.
- Minimax and Maximin Principle
- Importance and application of saddle point.

Slide 1



Slide 2

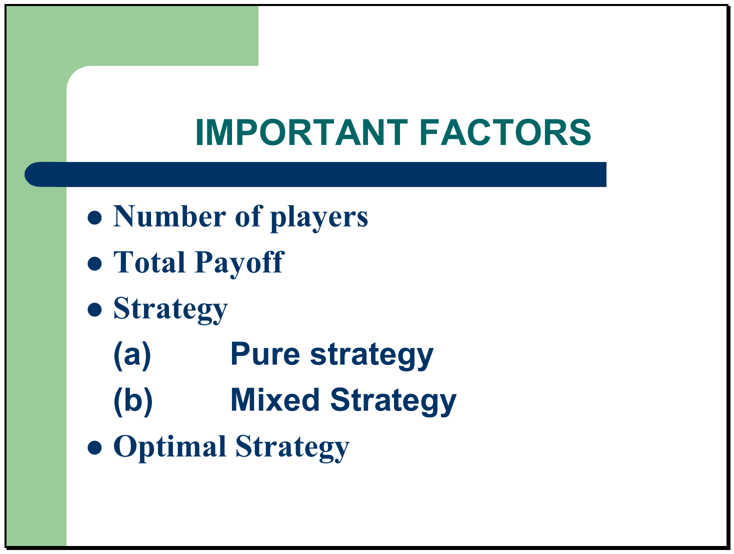

INTRODUCTION
Game theory is the study of how optimal strategies are formulated in conflict.

Slide 3

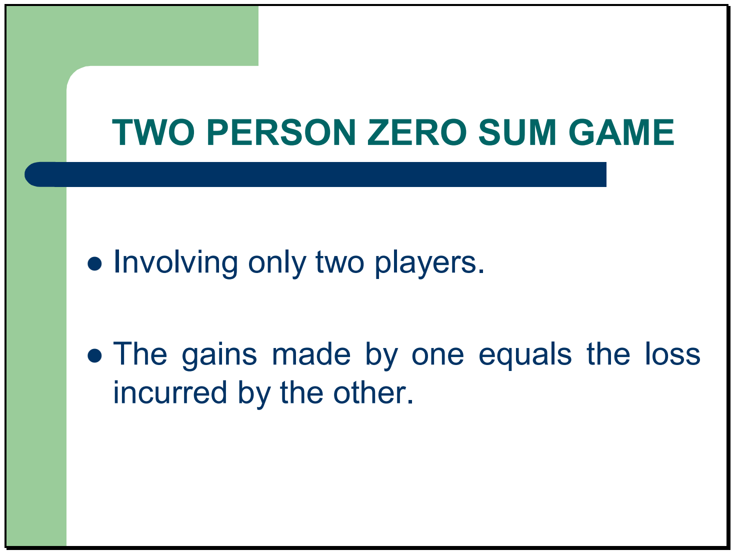
WHAT IS A GAME?

- A game refers to a situation in which two or more players are competing having different goals or objectives.

Slide 4


IMPORTANT FACTORS
● Number of players
● Total Payoff
● Strategy
(a) Pure strategy
(b) Mixed Strategy
● Optimal Strategy

Slide 5


TWO PERSON ZERO SUM GAME
Involving only two players.
The gains made by one equals the loss incurred by the other.

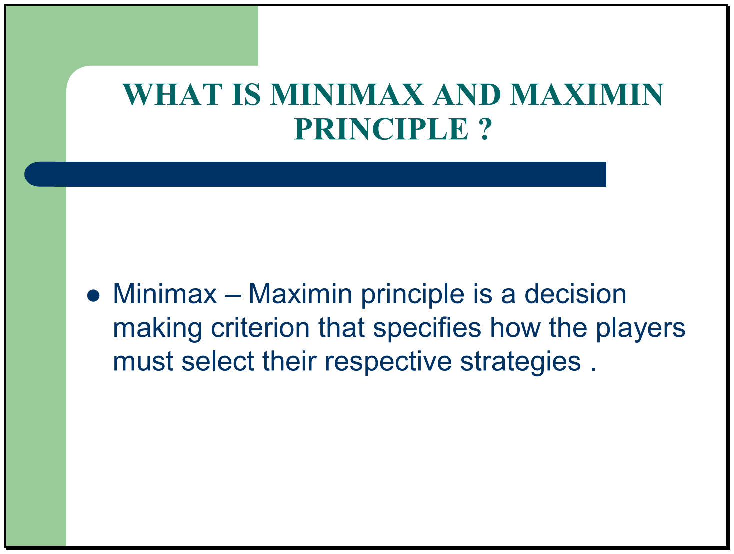
Slide 6

TWO PERSON ZERO SUM GAME PAYOFF MATRIX

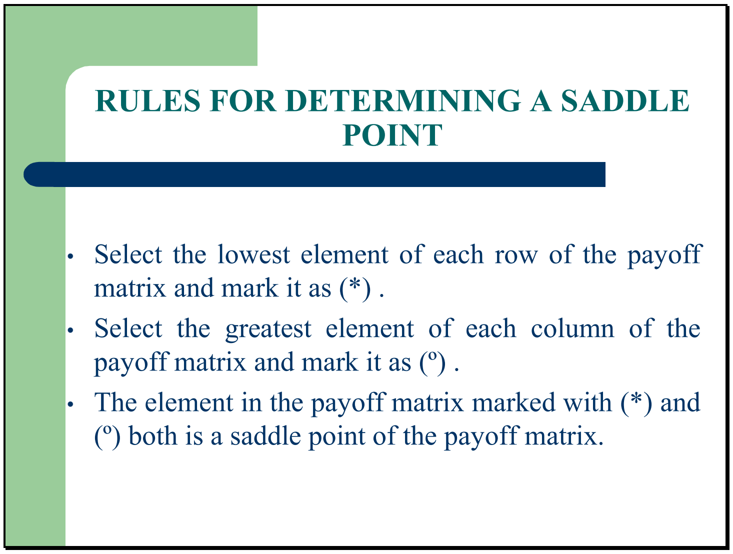
Player A's Strategies | **Player B's Strategies**

$$
\begin{array}{c}
 \\
A_1 \\
A_2 \\
\vdots \\
A_m
\end{array}
\quad
\begin{array}{cccc}
B_1 & B_2 & & B_n \\
\left\| \begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array} \right\|
\end{array}
$$

Slide 7


WHAT IS MINIMAX AND MAXIMIN PRINCIPLE ?
● Minimax – Maximin principle is a decision making criterion that specifies how the players must select their respective strategies .

Slide 8


RULES FOR DETERMINING A SADDLE POINT
• Select the lowest element of each row of the payoff matrix and mark it as (*) .
• Select the greatest element of each column of the payoff matrix and mark it as (º) .
• The element in the payoff matrix marked with (*) and (º) both is a saddle point of the payoff matrix.