UNIT 2
QUEUING THEORY

LESSON 24

Learning Objective:

- **Apply formulae to find solution that will predict the behaviour of the single server model II.**
- **Apply formulae to find solution that will predict the behaviour of the single server model III.**
- **Apply formulae to find solution that will predict the behaviour of the single server model IV**

Hello students,
In this lesson you are going to study about three more types of single server models that differ from single server model I in terms of their queue discipline.

Single server model II

{(M/M/1) : (∞ / SIRO)} model

This model is identical to the Model I with a difference only in queue discipline. In this model the selection of the customers is made in random order. Since the derivation of Pn is independent of any specific queue discipline, therefore in this model also we have

$Pn = (1-\rho)\,\rho^n$

N = 1, 2, 3, ……

Consequently, other performance measures will also remain unchanged in any queuing system as long as Pn remains unchanged.

Single server model III

{(M/ M/1) : (N / FCFS)}-- Exponential service; Limited queue

Suppose that no more than N customers can be accommodated at any time in the system due to certain reasons.
For example, a finite queue may arise due to physical constraint such as emergency room in a hospital; a clinic with certain number of chairs for waiting patients, etc.

Assumptions of this model are same as that of Model I except that the length of the queue be limited.
In this case the service rate does not have to exceed arrival rate in order to obtain steady state equations.

Therefore,

The probability of a customer in the system for n = 0, 1, 2, …., N are obtained as follows:

$$P_n = (\lambda / \mu)^n P_o \qquad ; n \le N$$

The steady_state solution in this case exists even for $\rho > 1$. This is due to the limited capacity of the system which controls the arrivals by the queue length (= *N* - 1) not by the relative rates of arrival, departure, λ and μ. If $\lambda < \mu$ and $N \rightarrow \infty$, then $P_n = (1 - \lambda / \mu)(\lambda / \mu)^n$, which is same as in Model I.

Performance Measures for Model III

- Expected number of customers in the system

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n=1}^{N} n \frac{(1 - \lambda/\mu)}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^n$$

$$= \sum_{n=1}^{N} n \frac{(1 - \rho)}{(1 - \rho^{N+1})} \rho^n$$

$$= \frac{(1-\rho)}{(1-\rho^{N+1})} \sum_{n=0}^{N} n\rho^n$$

$$= \frac{(1-\rho)}{(1-\rho^{N+1})} (\rho + 2\rho^2 + 3\rho^3 + \dots + N\rho^n)$$

- Expected queue length *or* expected number of customers waiting in the system

$$Lq = Ls - \lambda/\mu$$

$$= Ls - \lambda(1 - Pn) / \mu$$

- Expected waiting time of a customer in the system (waiting + service)

$$Ws = \frac{Ls}{\lambda(1-Pn)}$$

- Expected waiting time of a customer in the queue

$$Wq = Ws - 1/\mu \quad \text{or} \quad \frac{Lq}{\lambda(1-P_N)}$$

- Fraction of potential customers lost (= fraction of time system is full)

$$P_N = P_o \rho^N$$

Example

Consider a single server queuing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour and the maximum permissible calling units in the system is two. Derive the steady-state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.

Solution

From the data of the problem, we have

$$\lambda = 3 \text{ units per hour;}$$

$$\mu = 4 \text{ units per hour, and } N = 2$$

Then traffic intensity, $p = \lambda / \mu = 3/4 = 0.75$

The steady-state probability distribution of the number of *n* customers (calling units) in the system is

$$Pn = \frac{(1 - \rho)\rho^n}{1 - \rho^{N+1}} \quad ; \quad \rho \neq 1$$

$$= \frac{(1-0.75)(0.75)^n}{1 - (0.75)^{2+1}} \quad = \quad (0.43)(0.75)^n$$

and the expected number of calling units in the system is given by

$$Ls \quad = \quad \sum_{n=1}^{N} nP_n \quad = \quad \sum_{n=1}^{2} n(0.43)(0.75)^n$$

$$= \quad 0.43 \sum_{n=1}^{2} n(0.75)^n$$

$$= \quad 0.43 \;\{(0.75) + 2(0.75)^2 \; {}_3 (0.75)^3 + \ldots + N(0.75)^n)\}$$
$$= \quad 0.81$$

Try some problems yourself

Problem 1. Consider a single server queuing system with Poisson input, exponential'" service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible number calling units in the system is two. Derive the steady-state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.

Problem 2. If for a period of 2 hours in the day (8 to 10 a.m.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period: *(a)* the probability that the yard is empty, *(b)* average number of trains in the system; on the assumption that the line capacity of the yard is limited to 4 trains only.

Problem 3. Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.

(i) Find the effective arrival rate at the clinic.
(ii) What is the probability that an arriving patient will not wait?
(iii) What is the expected waiting time until a patient is discharged from the clinic?

Problem 4. In a car-wash service facility, cars arrive for service according to a Poisson distribution with mean 5 per hour. The time for washing and cleaning each car varies but is found to follow an exponential distribution with mean 10 minutes per car. The facility cannot handle more than one car at a time and has a total of 5 parking spaces.
(i) Find the effective arrival rate.
(ii) What is the probability that an arriving car will get service immediately upon arrival?
(iii) Find the expected number of parking spaces occupied.

Problem 5. A petrol station has a single pump and space for not more than 3 cars (2 waiting, 1 being served). A car arriving when the space is filled to capacity goes elsewhere for petrol. Cars arrive according to a Poisson distribution at a mean rate of one every 8 minutes. Their service time has an exponential distribution with a mean of 4 minutes.

The proprietor has the opportunity of renting an adjacent piece of land, which would provide space for an additional car to wait (He cannot build another pump.) The rent would be Rs 10 per week. The expected net profit from each customer is Re. 0.50 and the station is open 10 hours every day. Would it be profitable to rent the additional space?

Problem 6. If for a period of 2 hours in the day (8 to 10 a.m.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period

(a) the probability that the yard is empty, and
(b) the average number of trains in the system, on the assumption that the line capacity of the yard is limited to 4 trains only.

Problem 7. At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady-state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

Problem 8. Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate per hour.

i Find the effective arrival rate at the clinic.
ii What is the probability that an arriving patient will not wait? Will he find a vacant seat in the room?
iii What is the expected waiting time until a patient is discharged from the clinic?

Problem 9. Assume that goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time (there being 10 lines one of which is reserved for shunting purpose). Calculate the probability that the yard is empty and find the average queue length.

Problem 10. A petrol station has a single pump and space for not more than 3 cars (2 waiting, 1 being served). A car arriving when the space is filled to capacity goes elsewhere for petrol. Cars arrive according to a Poisson distribution at a mean rate of one every 8 minutes. Their service time has an exponential distribution with a mean of 4 minutes.

The owner has the opportunity of renting an adjacent piece of land, which would provide space for an additional car to wait. (He cannot build another pump.) The rent would be Rs 2000 per month. The expected net profit from each customer is Rs 2 and the station is open 10 hours everyday. Would it be profitable to rent the additional space?

SINGLE SERVER MODEL IV

{ (M/M/1) : (M/GD) } Single Server – Finite Population (source) of arrivals

This model is different from Model I in the sense that calling population is limited, say M. Thus arrival of additional customers is not allowed to join the system when the system becomes busy in serving the existing customers in the queue.

Few applications of this model are:

(i) A fleet of office cars available for 5 senior executives. Here these 5 executives are the customers, and the cars in the fleet are the servers.

(ii) A maintenance staff provides repair to M machines in a workshop. Here the M machines are customers and the repair staff members are the servers.

When there are n customers in the system, then system is left with the capacity to accommodate M-n more customers. Thus, further arrival rate of customers to the system will be λ (M-n). That is, for s=1, the arrival rate and service rate is stated as follows:

$$\lambda_n = \begin{cases} \lambda (M-n) & ; n=1,2.., M \\ 0 & ; n> N \end{cases}$$

$$\mu_n = \mu \qquad ; n=1,2,\ldots,M$$

Performance Measures of Model IV

1. Probability that the system is idle

$$P_o = \left(\sum_{N=0}^{M} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu} \right)^n \right)^{-1}$$

2. Probability that there are n customers in the system

$$P_n = \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n P_o \qquad ; n=1,2,\ldots, M$$

3 Expected number of customers in the queue (or queue length)

$$L_q = \sum_{n=1}^{M} (n-1)\ P_n = M - \left(\frac{\lambda + \mu}{\lambda}\right)(1-P_o)$$

4 Expected number of customers in the system

$$L_s = \sum_{n=0}^{M} nP_n = L_q + (1-P_o)$$

$$= M - \frac{\mu}{\lambda}(1-P_o)$$

5 Expected waiting time of a customer in the system

$$W_q = \frac{L_q}{\lambda\,(M-L_s)}$$

6. Expected waiting time of a customer in the system

$$Ws = Wq + \frac{1}{\mu} \text{ or } \frac{Ls}{\lambda (M-Ls)}$$

Example

A mechanic repairs four machines. The mean time between service requirements is 5 hours for each machine and forms an exponential distribution. The mean repair time is one hour and also follows the same distribution pattern.

Determine the following:

(a) Probability that the service facility will be idle,

(b) Probability of various number of machines (0 through 4) to be out of order and being repaired,

(c) Expected number of machines waiting to be repaired, and being repaired,

Would it be economical to engage two mechanics, each repairing only two machines?

Solution:

λ = 1/5 =0.2 machine/hour
μ = 1 machine /hour
M = 4 machines
$\rho = \lambda/\mu$ 0.2

(a) probability that the system shall be idle is

$$Po = \left(\sum_{n=0}^{M} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu} \right)^{n} \right)^{-1}$$

$$= \left(\sum_{n=0}^{M} \frac{4!}{(4-n)!} (0.2)^{n} \right)^{-1}$$

$$= \left(1 + \frac{4!}{3!} (0.2) + \frac{4!}{2!}(0.2)^2 + \frac{4!}{1!} (0.2)^3 + \frac{4!}{0!} (0.2)^4 \right)^{-1}$$

$= \quad [\ 1+4(0.2) + 4 \text{ x}3\ (0.04) + (4\text{x}3\text{x}2\ )\ (0.008) + (4\text{x}3\text{x}2\text{x}1)\ (0.00016\ ]^{-1}$

$= \quad [\ 1+0.8 + 0.48 + 0.192 + 0.000384]^{-1}$

$= \quad (\ 2.481)^{-1} = 0.4030.$

(b) Probability that there shall be various number of machines (0 through 4) in the system

$$P_n = \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu} \right) P_o \qquad ; n \le M$$

(c) The expected number of machines to be out of order and being repaired

$$L_s = M - \frac{\mu}{\lambda} (1 - P_o) \quad = 4 - \frac{1}{0.2} (1-0.403)$$

$$= 4 - 2.985 = 1.015 \text{ machines}$$

Calculation of P_n

N	$\frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n$	Probability
(1)	**(2)**	**(3) = (2) x Po**
0	1.00	0.4030
1	0.80	0.3224
2	0.48	0.1934
3	0.19	0.0765
4	0.00	0.0000

(The sum total of these probabilities is 0.9953 instead of 1. It is because of the approximation error.)

(d) Expected time a machine will wait in queue to be repaired

$$W_q = \frac{1}{\mu}\left(\frac{M}{1-Po} - \frac{\lambda+\mu}{\lambda}\right) = \left(\frac{4}{1-0.403} - \frac{0.2+1}{0.2}\right)$$

$$= \frac{4}{0.597} - 6 = 0.70 \text{ hours or 42 minutes}$$

(e) If there are two mechanics each serving two machines, then M = 2, and therefore,

$$Po = \left(\sum_{n=0}^{2} \frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1}$$

$$= [1+2(0.2) + 2 \text{ x } 1(0.2)^2] = 0.68$$

So, now let us summarise today's discussion:

Summary

We have discussed in details about

- Single server model II.
- Single server model III.
- Single server model IV
- Performance Measures of Single server model III, IV