Unit 1

Lesson 13: Sensitivity Analysis

Learning Objectives

- **What is Sensitivity Analysis ?**
- **Role of sensitivity analysis in Linear programming.**

Finding the optimal solution to a linear programming model is important, but it is not the only information available. There is a tremendous amount of *sensitivity information*, or information about what happens when data values are changed.

Recall that in order to formulate a problem as a linear program, you had to invoke a *certainty assumption*: you had to know what value the data took on, and you have made decisions based on that data. Often this assumption is somewhat dubious: the data might be unknown, or guessed at, or otherwise inaccurate. How can you determine the effect on the optimal decisions if the values change? Clearly some numbers in the data are more important than others. Can you find the ``important'' numbers?

Can you determine the effect of misestimation?

Linear programming offers extensive capability for addressing these questions. In this lecture I will show you how data changes show up in the optimal table. I am giving you two examples of how to interpret Solver's extensive output.

Sensitivity Analysis

Suppose you solve a linear program ``by hand'' ending up with an optimal table (or tableau to use the technical term). You know what an optimal tableau looks like: it has all non-negative values in Row 0 (which we will often refer to as the *cost row*), all non-negative right-hand-side values, and a basis (identity matrix) embedded. To determine the effect of a change in the data, I will try to determine how that change effected the final tableau, and try to reform the final tableau accordingly.

Cost Changes

The first change I will consider is changing a cost value by Δin the original problem. I am given the original problem and an optimal tableau. If you had done exactly the same calculations beginning with the modified problem, you would have had the same final tableau except that the corresponding cost entry would be Δlower (this is because you never do anything except add or subtract scalar multiples of Rows 1 through *m* to other rows; you never add or subtract Row 0 to other rows). For example, take the problem

Example 1

Max 3x+2y
Subject to
  x+y <= 4
  2x+y <= 6
  x,y >= 0

The optimal tableau to this problem (after adding $s_1$and $s_2$as slacks to place in standard form) is:

z	x	y	s_1	s_2	RHS
1	0	0	1	1	10
0	0	1	2	-1	2
0	1	0	-1	1	2

Suppose the cost for *x* is changed to $3+\Delta$in the original formulation, from its previous value 3. After doing the same operations as before, that is the same pivots, you would end up with the tableau:

z	x	y	s_1	s_2	RHS
1	$-\Delta$	0	1	1	10
0	0	1	2	-1	2
0	1	0	-1	1	2

Now this is not the optimal tableau: it does not have a correct basis (look at the column of *x*). But you can make it correct in form while keeping the same basic variables by adding Δ times the last row to the cost row. This gives the tableau:

z	x	y	s_1	s_2	RHS
1	0	0	$1-\Delta$	$1+\Delta$	$10+2\Delta$
0	0	1	2	-1	2
0	1	0	-1	1	2

Note that this tableau has the same basic variables and the same variable values (except for *z*) that your previous solution had.

Does this represent an optimal solution? It does only if the cost row is all non-negative. This is true only if

$$1-\Delta \geq 0$$

$$1+\Delta \geq 0$$

which holds for $-1 \leq \Delta \leq 1$. For any Δ in that range, our previous basis (and variable values) is optimal. The objective changes to $10+2\Delta$.

In the previous example, we changed the cost of a basic variable. Let's go through another example. This example will show what happens when the cost of a nonbasic variable changes.

Example2

Max 3x+2y + 2.5w
Subject to
x+y +2w <= 4
2x+y +2w <= 6
x,y,w >= 0

Here, the optimal tableau is :

z	x	y	w	s_1	s_2	RHS
1	0	0	1.5	1	1	10
0	0	1	2	2	-1	2
0	1	0	0	-1	1	2

Now suppose I change the cost on *w* from 2.5 to $2.5+\Delta$ in the formulation. Doing the same calculations as before will result in the tableau:

z	x	y	w	s_1	s_2	RHS
1	0	0	$1.5-\Delta$	1	1	10
0	0	1	2	2	-1	2
0	1	0	0	-1	1	2

In this case,I already have a valid tableau. This will represent an optimal solution if $1.5-\Delta \geq 0$, so $\Delta \leq 1.5$. As long as the objective coefficient of *w* is no more than 2.5+1.5=4 in the original formulation, my solution of *x*=2,*y*=2 will remain optimal.

The value in the cost row in the simplex tableau is called the *reduced cost*. It is zero for a basic variable and, in an optimal tableau, it is non-negative for all other variables (for a maximization problem).

Summary: Changing objective function values in the original formulation will result in a changed cost row in the final tableau. It might be necessary to add a multiple of a row to the cost row to keep the form of the basis. The resulting analysis depends only on keeping the cost row non-negative.

Right Hand Side Changes

For these types of changes, concentrate on maximization problems with all $\leq$ constraints. Other cases are handled similarly.

Take the following problem:

Example 3

$$\begin{aligned} \text{Max} \quad & 4x+5y \\ \text{Subject to} & \\ 2x+3y &<= 12 \\ x+y &<= 5 \\ x,y &>= 0 \end{aligned} \tag{1.1}$$

The optimal tableau, after adding slacks $s_1$and $s_2$is

z	x	y	s_1	s_2	RHS
1	0	0	1	2	22
0	0	1	1	−2	2
0	1	0	−1	3	3

Now suppose instead of 12 units in the first constraint, I only had 11. This is *equivalent* to forcing s_1 to take on value 1. Writing the constraints in the optimal tableau long-hand, we get

$$z + s_1 + 2s_2 = 22$$

$$y + s_1 - 2s_2 = 2$$

$$x - s_1 + 3s_2 = 3$$

If I force s_1 to 1 and keep s_2 at zero (as a nonbasic variable should be), the new solution would be *z* = 21, *y*=1, *x*=4. Since all variables are nonnegative, this is the optimal solution.

In general, changing the amount of the right-hand-side from 12 to $12 + \Delta$ in the first constraint changes the tableau to:

z	x	y	s_1	s_2	RHS
1	0	0	1	2	$22 + \Delta$
0	0	1	1	−2	$2 + \Delta$
0	1	0	−1	3	$3 - \Delta$

This represents an optimal tableau as long as the righthand side is all non-negative. In other words, I need Δ between -2 and 3 in order for the basis not to change. For any Δ in that range, the optimal objective will be $22 + \Delta$. For example, with Δ equals 2, the new objective is 24 with *y*=4 and *x*=1.

Similarly, if I change the right-hand-side of the second constraint from 5 to $5 + \Delta$ in the original formulation, we get an objective of $22 + 2\Delta$ in the final tableau, as long as $-1 \leq \Delta \leq 1$.

Perhaps the most important concept in sensitivity analysis is the *shadow price* λ_i^* of a constraint: *If the RHS of Constraint i changes by* Δ *in the original formulation, the optimal objective value changes by* $\lambda_i^* \Delta$. The shadow price λ_i^* can be found in the optimal tableau. It is the reduced cost of the slack variable s_i. So it is found in the cost row (Row 0) in the column corresponding the slack for Constraint i. In this case, $\lambda_1^* = 1$ (found in Row 0 in the column of s_1) and $\lambda_2^* = 2$ (found in Row 0 in the column of s_2). The value λ_i^* is really the marginal value of the resource associated with Constraint i. For example, the optimal objective value (currently 22) would increase by 2 if I could increase the RHS of the second constraint by $\Delta = 1$. In other words, the marginal value of that resource is 2, i.e. you are willing to pay up to 2 to increase the right hand side of the second constraint by 1 unit. You may have noticed the similarity of interpretation between shadow prices in linear programming and Lagrange multipliers in constrained optimization. Is this just a coincidence? Of course not. This parallel should not be too surprising since, after all, linear programming is a special case of constrained optimization. To derive this equivalence (between shadow prices and optimal Lagrange multipiers), one could write the KKT conditions for the linear program...but we will skip this in this course!

In summary, changing the right-hand-side of a constraint is identical to setting the corresponding slack variable to some value. This gives you the shadow price (which equals the reduced cost for the corresponding slack) and the ranges.

New Variable

The shadow prices can be used to determine the effect of a new variable (like a new product in a production linear program). Suppose that, in formulation (1.1), a new variable *w* has coefficient 4 in the first constraint and 3 in the second.

What objective coefficient must it have to be considered for adding to the basis?

If you look at making *w* positive, then this is equivalent to decreasing the right hand side of the first constraint by 4*w* and the right hand side of the second constraint by 3*w* in the original formulation. We obtain the same effect by making $s_1 = 4w$ and $s_2 = 3w$. The overall effect of this is to decrease the objective by $\lambda_1^*(4w) + \lambda_2^*(3w) = 1(4w) + 2(3w) = 10w$. The objective value must be sufficient to offset this, so the objective coefficient must be more than 10 (exactly 10 would lead to an alternative optimal solution with no change in objective).

Example 4

maximise
$3x_1 + 7x_2 + 4x_3 + 9x_4$
subject to
$x_1 + 4x_2 + 5x_3 + 8x_4 <= 9$ (1)
$x_1 + 2x_2 + 6x_3 + 4x_4 <= 7$ (2)
$x_i >= 0$ i=1,2,3,4

Solve this linear program using the simplex method.

- what are the values of the variables in the optimal solution?
- what is the optimal objective function value?
- which constraints are tight?
- what would you estimate the objective function would change to if:
    - we change the right-hand side of constraint (1) to 10
    - we change the right-hand side of constraint (2) to 6.5
    - we add to the linear program the constraint $x_3 = 0.7$