Unit 1

Lesson 12: A **Presentation on Duality Theorem**

1

Duality Theory

- The duality theory deals with the relationship between original and dual LP problems.

- It is very important in optimization and other areas of applied mathematics.

2

Duality Theory

- For the Simplex Method so far, we have only considered maximisation problems, with $\leq$ functional constraints.
- We will now consider minimisation problems with $\geq$ functional constraints.
- We still require all variables to be non-negative.
- We shall use Duality Theory for this purpose.

3

Duality Theory

Consider the following LP problem. Observe that here we use min rather than max, and $\geq$ rather than $\leq$ for the functional constraints.

$$\begin{aligned} \min\ C &= 5x_1 + 3x_2 \\ s.t. \quad x_1 + 3x_2 &\geq 8 \\ 2x_1 - 4x_2 &\geq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Since these "violate" the format we used so far, we cannot use our Simplex Method to solve this problem directly. We shall solve it indirectly via Duality Theory.

4

Duality Theory

Dual Problem: Given a (min, $\geq$) problem, we create a (max, $\leq$) problem by the following transformations:

- min ➔ max
- $\geq$ ➔ $\leq$ in the functional constraints
- RHS ➔ Objective function coefficients
- Objective function coefficients ➔ RHS
- Constraint coefficients are transposed

Transpose: Interchange rows and columns of a matrix, table, etc.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

5

Duality Theory

Constructing the dual problem of the primal problem.

Problem P (Primal)

$$\begin{aligned} \underline{\min}\ C &= 5x_1 + 3x_2 \\ s.t. \quad x_1 + 3x_2 &\ \underline{\geq}\ 8 \\ 2x_1 - 4x_2 &\ \underline{\geq}\ 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Problem D (Dual)

$$\begin{aligned} \underline{\max}\ P &= 8y_1 + 7y_2 \\ s.t. \quad y_1 + 2y_2 &\ \underline{\leq}\ 5 \\ 3y_1 - 4y_2 &\ \underline{\leq}\ 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

6

Duality Theory

Minimisation Problem in canonical form:

$$\min \quad C = b_1 x_1 + b_2 x_2 + \Lambda + b_m x_m$$

s.t.

m variables

n functional constraints

$$a_{11} x_1 + a_{12} x_2 + \Lambda + a_{1m} x_m \geq c_1$$
$$a_{21} x_1 + a_{22} x_2 + \Lambda + a_{2m} x_m \geq c_2$$
$$\cdots\cdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \Lambda + a_{nm} x_m \geq c_n$$
$$x_1, x_2, \Lambda, x_m \geq 0$$

7

Duality Theory

Dual Problem:

$$\max \quad P = c_1 y_1 + c_2 y_2 + \Lambda + c_n y_n$$

s.t.

n variables

m functional constraints

$$a_{11} y_1 + a_{21} y_2 + \Lambda + a_{n1} y_n \leq b_1$$
$$a_{12} y_1 + a_{22} y_2 + \Lambda + a_{n2} y_n \leq b_2$$
$$\cdots\cdots$$
$$a_{1m} y_1 + a_{2m} y_2 + \Lambda + a_{nm} y_n \leq b_m$$
$$y_1, y_2, \Lambda, y_n \geq 0$$

8

Duality Theory

Example 3.2 (see also Ex.1, Section 5-5, B & Z)

Construct the dual of the following problem:

Minimise: $C = 2x_1 + 8x_2 + 3x_3$

$$\text{s.t.} \quad \begin{aligned} x_1 + 2x_2 + 4x_3 &\geq 6 \\ x_1 + 3x_2 - 5x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual Problem: Maximize $P = 6y_1 + 8y_2$

$$\text{s.t.} \quad \begin{aligned} y_1 + y_2 &\leq 2 \\ 2y_1 + 3y_2 &\leq 8 \\ 4y_1 - 5y_2 &\leq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

9

Duality Theorem

- A LP problem has an optimal solution if and only if its dual has an optimal solution.
- If an optimal solution exists, then the optimal values of the objective functions of the two problems are the same. That is,

$$\text{Max } P = \text{Min } C$$

Good News: This theorem is constructive in that it gives a simple recipe for obtaining an optimal solution for the dual problem from the final simplex tableau of the original problem and vice versa.

10

Duality Theory

Recipe

How can we obtain an optimal solution to a (min, ≥) problem from the final simplex tableau of its dual (max, ≤) problem ?

- Solve the dual (max, ≤) problem using the Simplex Method.
- Record the entries of the slack variables in the last row of the final simplex tableau.
- These coefficients are equal to the optimal values of the respective dual variables.

11

Duality Theory

Example 3.3 (see also Ex.2, Section 5-5, B & Z)

Solve the following problem by first constructing its dual, and then using the simplex method on the dual.

Minimise: $C = x_1 + 2x_2$

subject to:

$$x_1 + 0.5x_2 \geq 2$$
$$x_1 - x_2 \geq 2$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

12

Duality Theory

The dual problem is as follows:

Maximize: $P = 2y_1 + 2y_2 + 3y_3$

subject to:

$$y_1 + y_2 + y_3 \le 1$$
$$0.5y_1 - y_2 + y_3 \le 2$$
$$y_1, y_2, y_3 \ge 0$$

Adding slack variables:

Maximize: $P = 2y_1 + 2y_2 + 3y_3$

subject to:

$$y_1 + y_2 + y_3 + s_1 \qquad = 1$$
$$0.5y_1 - y_2 + y_3 + \qquad s_2 = 2$$
$$y_1, y_2, y_3, s_1, s_2 \ge 0$$

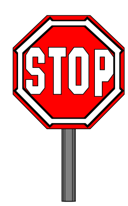
13

Duality Theory

Initial Simplex Tableau

BV	y1	y2	y3	s1	s2	P	RHS	
s1	1	1	1	1	0	0	1	R1 → R1
s2	1/2	–1	1	0	1	0	2	R2 – R1 → R1
P	–2	–2	–3	0	0	1	0	R3 +3R1 → R1

BV	y1	y2	y3	s1	s2	P	RHS
y3	1	1	1	1	0	0	1
s2	–1/2	–2	0	–1	1	0	1
P	1	1	0	3	0	1	3



Recipe: optimal solution of the (min, ≥) problem = entries of slack variables in the last row of the final tableau of the dual (max, ≤) problem.

Duality Theory

Report:

- The optimal solution to the dual (max, ≤) problem is $(y_1, y_2, y_3) = (0, 0, 1)$.
- The optimal solution to the original (min, ≥) problem is $(x_1, x_2) = (3, 0)$.
- The optimal values of the objective functions are:

$$\text{Max } P = \text{Min } C = 3$$

- Do Questions 1-4, Example Sheet 4.
- Look at (no need to do unless you want to) the application problems at the end of Sections 5-5, 5-6 and Review, to get an idea of some applications of LP.
- Do Example 3.5 graphically to check the answer.