

# Unit 1

## Lesson 11: Duality in linear programming

### Learning objectives:

- Introduction to dual programming.
- Formulation of Dual Problem.

### Introduction

For every LP formulation there exists another unique linear programming formulation called the 'Dual' (the original formulation is called the 'Primal'). The Dual formulation can be derived from the same data from which the primal was formulated. The Dual formulated can be solved in the same manner in which the Primal is solved since the Dual is also a LP formulation.

The Dual can be considered as the 'inverse' of the Primal in every respect. The column coefficients in the Primal constraints become the row co-efficients in the Dual constraints. The coefficients in the Primal objective function become the right hand side constraints in the Dual constraints. The column of constants on the right hand side of the Primal constraints becomes the row of coefficients of the dual objective function. The 'direction of the inequalities are reversed. If the primal objective function is a 'Maximization' function then the dual objective function is a 'Minimization' function and vice-versa.

The concept of duality is very much in useful to obtain additional information about the variation in the optimal solution when certain changes are effected in the constraint co-efficient, resource availabilities and objective function co-efficients. This is termed as post optimality or sensitivity analysis.

### Dual Problem Construction:

- If the primal is a maximization problem, then its dual is a minimization problem (and vice versa).
- Use the variable type of one problem to find the constraint type of the other problem.
- Use the constraint type of one problem to find the variable type of the other problem.
- The RHS elements of one problem become the objective function coefficients of the other problem (and vice versa).

- The matrix coefficients of the constraints of one problem is the transpose of the matrix coefficients of the constraints for the other problem. That is, rows of the matrix becomes columns and vice versa.

## Dual Formation

Following are the steps adopted to convert primal problem into its dual.

- Step 1. For each constraint in primal problem there is an associated variable in dual problem.
- Step 2. The elements of right hand side of the constraints will be taken as the co-efficient of the objective function in the dual problem.
- Step 3. If the primal problem is maximization, then its dual problem will be minimization and vice versa.
- Step 4. The inequalities of constraints should be interchanged from  $\geq$  to  $\leq$  and vice versa and the variables in both the problems and non-negative.
- Step 5. The row of primal problem are changed to columns in the dual problem. In other words the matrix A of the primal problem will be changed to its transpose (A) for the dual problem.
- Step 6. The co-efficient of the objective function will be taken the right hand side of the constraints of the dual problem.

## Problems and Solutions

An example will clarify the concept basis. Consider the following 'Primal' LP formulation

### Example 3.1,

MaXimize  $12x_1 + 10x_2$   
 subject to  $2x_1 + 3x_2 \leq 18$   
 $2x_1 + x_2 \leq 14$   
 $x_1, x_2 \geq 0$

### Solution 3.1,

The 'Dual' formulation for this problem would be

Minimize  $18Y_1 + 14Y_2$   
 subject to  $2Y_1 + 2Y_2 \geq 12$   
 $3Y_1 + Y_2 \geq 10$   
 $Y_1 \geq 0, Y_2 \geq 0$

Note the following:

1. The column coefficient in the Primal constraint namely (2,2) and (3,1) have become the row co-efficient in the Dual constraints.

2. The co-efficient of the Primal objective function namely, 12 and 10 have become the constants in the right hand side of the Dual constraints.
3. The constants of the Primal constraints, namely 18 and 14, have become the co-efficient in the Dual objective function.
4. The direction of the inequalities have been reserved. The Primal constraints have the inequalities of  $\leq$ ; while the Dual constraints have the inequalities of  $\geq$ .
5. While the Primal is a 'Maximization' problem the Dual is a 'Minimization' problem

Why construct the Dual formulation? This is for a number of reasons. The solution to the dual problem provides all essential information about the solution to the Primal problem. For an LP problem, the solution can be determined either by solving the original problem or the dual problem. Sometimes it may be easier to solve the Dual problem rather than the Primal problem. (For instance, when the primal involves few variables but many constraints.)

### Example 3.2,

Obtain the dual problem of the following primal formulation.

$$\text{Maximize } Z = 2X_1 + 5X_2 + 6X_3$$

$$\begin{array}{llll} \text{Subject to} & 5X_1 + 6X_2 - X_3 & \leq & 3 \\ & -2X_1 + X_2 + 4X_3 & \leq & 4 \\ & X_1 - 5X_2 + 3X_3 & \leq & 1 \\ & -3X_1 - 3X_2 + 7X_3 & \leq & 6 \\ & X_1, X_2, X_3 & \geq & 0 \end{array}$$

### Solution 3.2,

Step 1 : Write the objective function of the Dual. As there are four constraints in the primal, the objective function of the dual will have 4 variables.

$$\text{Minimize } r^* = 3W_1 + 4W_2 + W_3 + 6W_4$$

Step 2 : Write the constraints of the dual. As all the constraints in the primal are ' $\leq$ ', the constraints in the dual will be ' $\geq$ '. The column coefficients of the primal become the row coefficients of the dual.

$$\begin{array}{l} \text{Constraints. } 5W_1 - 2W_2 + W_3 - 3W_4 \geq 2 \\ \quad \quad \quad 6W_1 + W_2 - 5W_3 - W_4 \geq 5 \\ \quad \quad \quad -W_1 + 4W_2 + 3W_3 + 7W_4 \geq 6 \end{array}$$

Step 3 : Therefore the dual of the Primal is :

$$\text{Minimize } Z = 3W_1 + 4W_2 + W_3 + 6W_4$$

$$\begin{array}{rcl}
\text{Subject to : } & 5W_1 - 2W_2 + W_3 - 3W_4 & \geq 2 \\
& 6W_1 + W_2 - 5W_3 - 3W_4 & \geq 5 \\
& -W_1 + 4W_2 + 3W_3 + 7W_4 & \geq 6 \\
& W_1, W_2, W_3, W_4 & \geq 0
\end{array}$$

### Example 3.3,

Obtain the dual of the- following linear programming problem.

$$\text{Minimize } Z = 5X_1 - 6X_2 + 4X_3$$

Subject to the constraints :

$$\begin{array}{rcl}
3X_1 + 4X_2 + 6X_3 & \geq & 9 \\
X_1 + 3X_2 + 2X_3 & \geq & 5 \\
7X_1 - 2X_2 - X_3 & \geq & 10 \\
X_1 - 2X_2 + 4X_3 & \geq & 4 \\
2X_1 + 5X_2 - 3X_3 & \geq & 3 \\
X_1, X_2, X_3 & \geq & 0
\end{array}$$

### Solution 3.3,

In this problem one of the primal constraints (namely  $7X_1 - 2X_2 - X_3 \leq 10$ ) is a " $\leq$ ", constraints while all the other are " $\geq$ " constraints. The dual cannot be worked out unless all the constraints are in the same direction. To convert this into " $\geq$ " constraint, multiply both the sides of the equation by "-" sign. After multiplying the constraint by "-" sign, it will become  $7x_1 + 2x_2 + x_3 \geq -10$ . Now all the constraints are in the same direction and the dual can be worked out.

The dual formulation is :

$$\text{Maximize } Z = 9W_1 + 5W_2 - 10W_3 + 4W_4 + 3W_5$$

Subject to :

$$\begin{array}{rcl}
3W_1 + W_2 - 7W_3 + W_4 + 2W_5 & \leq & 5 \\
4W_1 + 3W_2 + 2W_3 - 2W_4 + 5W_5 & \leq & -6 \\
6W_1 + 2W_2 + W_3 + 4W_4 + -3W_5 & \leq & 4 \\
W_1, W_2, W_3, W_4, W_5 & \geq & 0
\end{array}$$

