Unit 1

Lesson 10: Two-Phase Simplex

Learning Objective:

- **Two-Phase Method to solve LPP**

So far, you have developed an algorithm to solve formulated linear programs (The Simplex Method). Notice that, your algorithm starts with an initial basic feasible solution and if all the inequalities of the constraints are of “less than or equal to” type, the origin is always our starting point.

For a “greater than or equal to” constraint, the slack variable in the equality form has a negative coefficient for the origin. Again, “equality” constraints have no slack variables. If either type of constraint is part of the model, there is no convenient initial basic feasible solution.

This is where Two-Phase Simplex comes in. It involves two phases (thus the name!):

Phase 1: Has the goal of finding a basic feasible solution,

Phase 2: Has the goal of finding the optimum solution.

The procedure of this technique is as follows:

Phase 1:

1.First, add nonnegative variables to the left hand side of the types “$\geq$” and “=”. These variables are called “*artificial variables*“.

2.Formulate a new problem by replacing the original objective function by the sum of the artificial variables and minimize it to ensure that the artificial variables will be zero in the final solution which gives you a basic feasible solution that you searched.

Phase 2:

1.If at the end of Phase 1, all artificial variables can be set to zero value, then a basic feasible solution is found and we can pass to the second phase.

In order to start the second phase, the objective function must be expressed in terms of the non basic variables only. After applying the proper transformations, proceed with the regular steps of the simplex method.

To show how a two phase method is applied, see an example.

Example 1

$$Max\ Z=2X_1+3X_2$$

s.t.

$$\begin{aligned}
-X_1+\ X_2 &\leq 5 \\
X_1+3X_2 &\leq 35 \\
X_1 \qquad &\leq 20 \\
X_1+\frac{3}{2}X_2 &\geq 10
\end{aligned}$$

and,

$$X_1,X_2 \geq 0.$$

First, the standard form of the problem can be converted from the canonical form as follows:

$$Max\ Z=2X_1+3X_2$$

s.t.

$$\begin{aligned}
-X_1+\ X_2+S_1 \qquad\qquad &= 5 \\
X_1+3X_2 \quad +S_2 \qquad &= 35 \\
X_1 \qquad\qquad +S_3 \quad &= 20 \\
X_1+\frac{3}{2}X_2 \qquad\qquad -S_4+a &= 10
\end{aligned}$$

and,

$$X_1,X_2 \geq 0.$$

where “a” is the artificial variable added as it is explained above. Now, it is time to go through Phase 1.

Phase 1: Implement the new problem of minimizing the sum of the artificial variables.

$$Min\ W = a$$

since there is on one artificial variable.

Because the artificial variable is the basic variable of the fourth equation, we should write W function in terms of non basic variables. That is,

$$Min\ W = 10 - \left(X_1 + \frac{3}{2} X_2 - S_4 \right).$$

Now, solve this by the simplex method.

Iteration 1.1

Basic	X_1	X_2	S_1	S_2	S_3	S_4	a	RHS
W	1	3/2	0	0	0	-1	0	10
S_1	-1	1	1	0	0	0	0	5
S_2	1	3	0	1	0	0	0	35
S_3	1	0	0	0	1	0	0	20
a	1	3/2	0	0	0	-1	1	10

In the table above, entering and leaving variables are selected as follows:

Iteration 1. 2

Basic	X_1	X_2	S_1	S_2	S_3	S_4	a	RHS	
W	1	3/2	0	0	0	-1	0	10	
S_1	-1	**1**	1	0	0	0	0	5	5
S_2	1	3	0	1	0	0	0	35	35/3
S_3	1	0	0	0	1	0	0	20	
a	1	3/2	0	0	0	-1	1	10	20/3

After applying the calculations, you will get

Iteration 1. 3

Basic	X_1	X_2	S_1	S_2	S_3	S_4	a	RHS
W	5/2	0	-3/2	0	0	-1	0	5/2
X_2	-1	1	1	0	0	0	0	5
S_2	4	0	-3	1	0	0	0	20
S_3	1	0	0	0	1	0	0	20
A	5/2	0	-3/2	0	0	-1	1	5/2

This table above is not optimum so you go on iterating.

Iteration 2.1

Basic	X_1	X_2	S_1	S_2	S_3	S_4	a	RHS	
W	5/2	0	-3/2	0	0	-1	0	5/2	
X_2	-1	1	1	0	0	0	0	5	
S_2	4	0	-3	1	0	0	0	20	5
S_3	1	0	0	0	1	0	0	20	20
A	**5/2**	0	-3/2	0	0	-1	1	5/2	1

Apply the calculations again and reach to the optimum W=0:

Iteration 2.2

Basic	X_1	X_2	S_1	S_2	S_3	S_4	a	RHS
W	0	0	0	0	0	0	-1	0

X_2	0	1	2/5	0	0	-2/5	0	6
S_2	0	0	-3/5	1	0	8/5	0	16
S_3	0	0	3/5	0	1	2/5	0	19
X_1	1	0	-3/5	0	0	-2/5	1	1

Now, you have an initial basic feasible solution, which is (X_1 , X_2) = (1,6). So, the phase 1 is completed and you can pass through phase 2.

<u>Phase 2:</u> The objective function Z must be expressed in terms of non basic variables. Our non basic variables are S_1 and S_4 and we will write the basic variables in the original equation in terms of them.

$$Max\ Z=2X_1+3X_2=2\left(1+\frac{3}{5}S_1+\frac{2}{5}S_4\right)+3\left(6-\frac{2}{5}S_1+\frac{2}{5}S_4\right)$$

$$\Rightarrow Z=20+2S_4$$

You can write the first table of phase 2 now. The last table in phase 1 is just copied with two exceptions. You delete the column of the artificial variable and change the W equation to Z equation.

<u>Iteration 3.1</u>

Basic	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	0	0	0	0	0	-2	20
X_2	0	1	2/5	0	0	-2/5	6
S_2	0	0	-3/5	1	0	8/5	16
S_3	0	0	3/5	0	1	2/5	19
X_1	1	0	-3/5	0	0	-2/5	1

Since this is a maximization problem, this table is not optimum and you have to apply simplex method to find the optimum.

Iteration 3.2

Basic	X_1	X_2	S_1	S_2	S_3	S_4	RHS	
Z	0	0	0	0	0	-2	20	
X_2	0	1	2/5	0	0	-2/5	6	
S_2	0	0	-3/5	1	0	**8/5**	16	10
S_3	0	0	3/5	0	1	2/5	19	95/2
X_1	1	0	-3/5	0	0	-2/5	1	

After determining the entering and leaving variables, you get the table 3.3

Iteration 3.3

Basic	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	0	0	-6/8	5/4	0	0	40
X_2	0	1	2/8	2/8	0	0	10
S_4	0	0	-3/8	5/8	0	1	10
S_3	0	0	6/8	-2/8	1	0	15
X_1	1	0	-6/8	2/8	0	0	5

you couldn't get to the optimum table yet, so you go on.

Iteration 3.4

Basic	X_1	X_2	S_1	S_2	S_3	S_4	RHS	
Z	0	0	-6/8	5/4	0	0	40	
X_2	0	1	2/8	2/8	0	0	10	40
S_4	0	0	-3/8	5/8	0	1	10	
S_3	0	0	**6/8**	-2/8	1	0	15	20
X_1	1	0	-6/8	2/8	0	0	5	

After the appropriate calculations, you will get to the following optimum table :

Iteration 4.1

Basic	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	0	0	0	1	1	0	55
X_2	0	1	0	1/3	-1/3	0	5
S_4	0	0	0	1/2	½	1	35/2
S_1	0	0	1	-1/3	4/3	0	20
X_1	1	0	0	2/8	1	0	20

That is, (X_1 , X_2) = (20,5) is the optimum point with optimum Z = 55.

Exercise1

$$\begin{array}{llll}
\text{minimize } z = & 2x_1 & +3x_2 & \\
\text{subject to} & x_1 & +2x_2 & \geq 10 \\
 & x_1 & +4x_2 & \geq 12 \\
 & 3x_1 & +2x_2 & \geq 15 \\
 & x_1 \geq 0, & x_2 \geq 0 &
\end{array}$$

Solution 1

Exercise 2

$$\begin{aligned} \text{minimize } z = & -x_1 - 2x_2 - x_3 \\ \text{subject to } & x_1 + 4x_2 - 2x_3 \geq 120 \\ & x_1 + x_2 + x_3 = 60 \\ & x_1 \geq 0,\ x_2 \geq 0\ x_3 \geq 0 \end{aligned}$$

Solution 2

Exercise 3

$$\begin{aligned}
\max \quad & z = 3x_1 + x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \geq 1 \\
& 2x_1 + x_2 \leq 4 \\
& x_1 + x_2 = 3 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Solution 3

Recap

Step 1: make all right-hand-side values nonnegative.

Step 2: put the LP in standard form.

Step 3: add an artificial variable to each row without a slack.

Step 4: write the initial dictionary, using slack and artificial variables as basic variables.

Step 5: replace the objective with minimizing the sum of all artificial variables (call it “w” to keep it separate from the original objective).

Recap

Step 6: write the new objective in terms of nonbasic variables.

Step 7: solve the Phase I LP. If w > 0, the original LP is infeasible.

Step 8: replace the original objective, in terms of nonbasic variables. Delete all nonbasic artificial variables. This creates the Phase II LP.

Step 9: Optimize the Phase II LP. As artificial variables become nonbasic, delete them.

Recap

- Three outcomes when solving Phase I LP:
  - Minimum w > 0: LP infeasible
  - Mimimum w = 0, all artificial variables nonbasic:
    - replace objective
    - delete artificial variables
    - continue with Phase II
  - Minimum w = 0, some artificial variable basic (and equal to zero):
    - replace objective
    - delete nonbasic artificial variables
    - continue with Phase II
    - delete each artificial variable as it becomes nonbasic.