Unit 1

Lesson 8: Special cases of LPP

Learning outcomes

SolvingSpecial cases of Linear Programming Problem using Simplex Method :

- **Alternate Optimal Solutions.**
- **Degeneracy.**
- **Unboudedness.**
- **Infeasibility.**

In the previous lecture we have learnt how to solve a linear program using simplex method. Properties of Linear Programs

There are three possible outcomes for a linear program: **it is infeasible, it has an unbounded optimum or it has an optimal solution.**

If there is an optimal solution, there is a *basic* optimal solution. Remember that the number of basic variables in a basic solution is equal to the number of constraints of the problem, say *m*. So, even if the total number of variables, say *n*, is greater than *m*, at most *m* of these variables can have a positive value in an optimal basic solution.

Today in this lecture we will study about Alternate Optimal Solutions, Degeneracy, Unboudedness, Infeasibility

Alternate Optimal Solutions

Let us solve a small example:

Example1

$$\begin{array}{llll} \max & x_1 & +\frac{1}{2}x_2 & \\ & 2x_1 & +x_2 & \leq 4 \\ & x_1 & +2x_2 & \leq 3 \\ & x_1 \geq 0, & x_2 \geq 0 & \end{array}$$

As before, we add slacks $x_3$and x_4, and we solve by the simplex method, using tableau representation.

z	x_1	x_2	x_3	x_4	RHS		Basic solution
1	−1	$-\frac{1}{2}$	0	0	0	basic	$x_3 = 4$ $x_4 = 3$
0	2	1	1	0	4	nonbasic	$x_1 = x_2 = 0$
0	1	2	0	1	3	$z = 0$	
1	0	0	$\frac{1}{2}$	0	2	basic	$x_1 = 2$ $x_4 = 1$
0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	nonbasic	$x_2 = x_3 = 0$
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	1	$z = 2$	

Now Rule 1 shows that this is an optimal solution. Interestingly, the coefficient of the nonbasic variable $x_2$in Row 0 happens to be equal to 0. Going back to the rationale that allowed us to derive Rule 1, we observe that, if we increase x_2(from its current value of 0), this will not effect the value of *z*. Increasing $x_2$produces changes in the other variables, of course, through the equations in Rows 1 and 2. In fact, we can use Rule 2 and pivot to get a different basic solution with the same objective value *z*=2.

z	x_1	x_2	x_3	x_4	RHS		Basic solution
1	0	0	$\frac{1}{2}$	0	2	basic	$x_1 = \frac{5}{3}$ $x_2 = \frac{2}{3}$
0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	nonbasic	$x_3 = x_4 = 0$
0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$z = 2$	

Note that the coefficient of the nonbasic variable $x_4$in Row 0 is equal to 0. Using $x_4$as entering variable and pivoting, we would recover the previous solution!

Degeneracy

Example2

Max Z = 2 $x_1 + x_2$

$3x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_2 \le 3$

$x_1 \ge 0$, $x_2 \ge 0$

Let us solve this problem using the -by now familiar- simplex method. In the initial tableau, we can choose $x_1$as the entering variable (Rule 1) and Row 2 as the pivot row (the minimum ratio in Rule 2 is a tie, and ties are broken arbitrarily). We pivot and this yields the second tableau below.

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution			
1	-2	-1	0	0	0	0	basic	$x_3 = 6$	$x_4 = 2$	$x_5 = 3$
0	3	1	1	0	0	6	nonbasic	$x_1 = x_2 = 0$		
0	1	-1	0	1	0	2	$z = 0$			
0	0	1	0	0	1	3				
1	0	-3	0	2	0	4	basic	$x_1 = 2$	$x_3 = 0$	$x_5 = 3$
0	0	4	1	-3	0	0	nonbasic	$x_2 = x_4 = 0$		
0	1	-1	0	1	0	2	$z = 4$			
0	0	1	0	0	1	3				

Note that this basic solution has a basic variable (namely x_3) which is equal to zero. When this occurs, we say that the basic solution is *degenerate*. Should this be of concern? Let us continue the steps of the simplex method. Rule 1 indicates that x_2 is the entering variable. Now let us apply Rule 2. The ratios to consider are $\frac{0}{4}$ in Row 1 and $\frac{3}{1}$ in Row 3. The minimum ratio occurs in Row 1, so let us perform the corresponding pivot.

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution			
1	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	0	4	basic	$x_1 = 2$	$x_2 = 0$	$x_5 = 3$
0	0	1	$\frac{1}{4}$	$-\frac{3}{4}$	0	0	nonbasic	$x_3 = x_4 = 0$		
0	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	2	$z = 4$			
0	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	3				

We get exactly the same solution! The only difference is that we have interchanged the names of a nonbasic variable with that of a degenerate basic variable (x_2 and x_3). Rule 1 tells us the solution is not optimal, so let us continue the steps of the simplex method. Variable x_4 is the entering variable and the last row wins the minimum ratio test. After pivoting, we get the tableau:

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution			
1	0	0	$\frac{2}{3}$	0	$\frac{1}{3}$	5	basic	$x_1 = 1$	$x_2 = 3$	$x_4 = 4$
0	0	1	0	0	1	3	nonbasic	$x_3 = x_5 = 0$		
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	1	$z = 5$			
0	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	4				

By Rule 1, this is the optimal solution. So, after all, degeneracy did not prevent the simplex method to find the optimal solution in this example. It just slowed things down a little. Unfortunately, on other examples, degeneracy may lead to *cycling*, i.e. a sequence of pivots that goes through the same tableaus and repeats itself indefinitely. In theory, cycling can be avoided by choosing the entering variable with smallest index in Rule 1, among all those with a negative coefficient in Row 0, and by breaking ties in the minimum ratio test by choosing the leaving variable with smallest index (this is known as Bland's rule). This rule, although it guaranties that cycling will never occur, turns out to be somewhat inefficient. Actually, in commercial codes, no effort is made to avoid cycling. This may come as a surprise, since degeneracy is a frequent occurence. But there are two reasons for this:

- Although degeneracy is frequent, cycling is extremely rare.
- The precision of computer arithmetic takes care of cycling by itself: round off errors accumulate and eventually gets the method out of cycling.

Our example of degeneracy is a 2-variable problem, so you might want to draw the constraint set in the plane and interpret degeneracy graphically.

Unbounded Optimum

Example 3 **Max** $Z = 2x_1 + x_2$

s.t $-x_1 + x_2 \leq 1$

$x_1 - 2x_2 \leq 2$

$x_1 \geq 0 \ , \ x_2 \geq 0$

Solving by the simplex method, we get:

z	x_1	x_2	x_3	x_4	RHS		Basic solution
1	-2	-1	0	0	0	basic	$x_3 = 1$ $x_4 = 2$
0	-1	1	1	0	1	nonbasic	$x_1 = x_2 = 0$
0	1	-2	0	1	2	$z = 0$	
1	0	-5	0	2	4	basic	$x_1 = 2$ $x_3 = 3$
0	0	-1	1	1	3	nonbasic	$x_2 = x_4 = 0$
0	1	-2	0	1	2	$z = 4$	

At this stage, Rule 1 chooses x_2 as the entering variable, but there is no ratio to compute, since there is no positive entry in the column of x_2. As we start increasing x_2, the value of *z* increases (from Row 0) and the values of the basic variables increase as well (from Rows 1 and 2). There is nothing to stop them going off to infinity. So the problem is unbounded.

So you have seen how the special cases are solved.