Unit 1

Lesson 5. : Special cases of LPP

Learning Outcomes

Special cases of linear programming problems

- **Alternative Optima**
- **Infeasible Solution**
- **Unboundedness**

In the previous lecture we have discussed some linear programming problems which may be called ‘ well behaved’ problems. In such cases, a solution was obtained, in some cases it took less effort while in some others it took a little more. But a solution was finally obtained.

a) Alternative Optima, b) Infeasible(or non existing) solution, c) unbounded solution.

First Special Cases

a)Alternative Optima

When the objective function is parallel to a binding constraint (a constraint that is satisfied in the equality sense by the optimal solution), the objective function will assume the same optimal value at more than one solution point. For this reason they are called **alternative optima.** The example 1 shows that normally there is infinity of such solutions. The example also demonstrates the practical significance of encountering alternative optima.

Example1

Maximize $z = 2x_1 + 4x_2$

Subject to

$x_1 + 2x_2 \leq 5$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

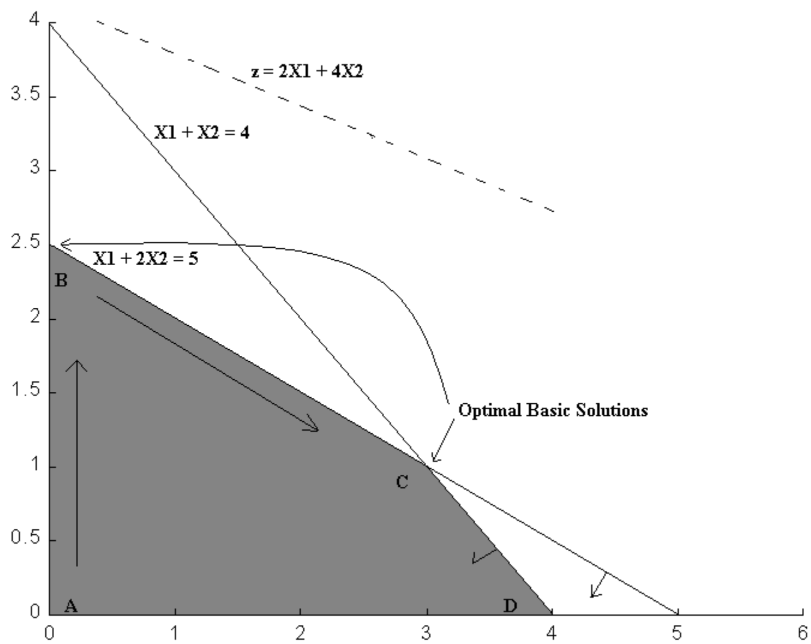


Figure demonstrates how alternative optima can arise in LP model when the objective function is parallel to a binding constraint. Any point on the line segment BC represents an alternative optimum with the same objective value z = 10. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of the points B and C. That is, given $0 \alpha 1$ and

B: $x_1 = 0, x_2 = 5/2$

C: $x_1 = 3$, $x_2 = 1$

Then all points on the line segment BC are given by

$x_1 = \alpha(0) + 3(1-\alpha) = 3 - 3\alpha$

$x_2 = \alpha(5/2) + 1(1-\alpha) = 1 + 3\alpha/2$

Observe that when $\alpha = 0$, $(x_1, x_2) = (3, 1)$, which is point C. When $\alpha = 1$, $(x_1, x_2) = (0, 5/2)$, which is point B. For values of α between 0 and 1, (x_1, x_2) lies between B and C.

In practice, knowledge of alternative optima is useful because it gives management the opportunity to choose the solution that best suits their situation without experiencing any deterioration in the objective value. In the example, for instance, the solution at B shows that only activity 2 is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, it may be advantageous from the standpoint of sales competition to produce two products rather than one. In this case the solution at C would be recommended.

b) Infeasible 2-var LP's

Consider again the original prototype example, modified by the additional requirements (imposed by the company's marketing department) that the daily production of product P_1 must be at least 30 units, and that of product P_2 should exceed 20 units. These requirements introduce two new constraints into the problem formulation, i.e.,

$$X_1 \geq 30$$

$$X_2 \geq 20$$

Attempting to plot the feasible region for this new problem, we get Figure 2, which indicates that there are no points on the (X_1, X_2)-plane that satisfy all constraints, and therefore our problem is *infeasible* (over-constrained).

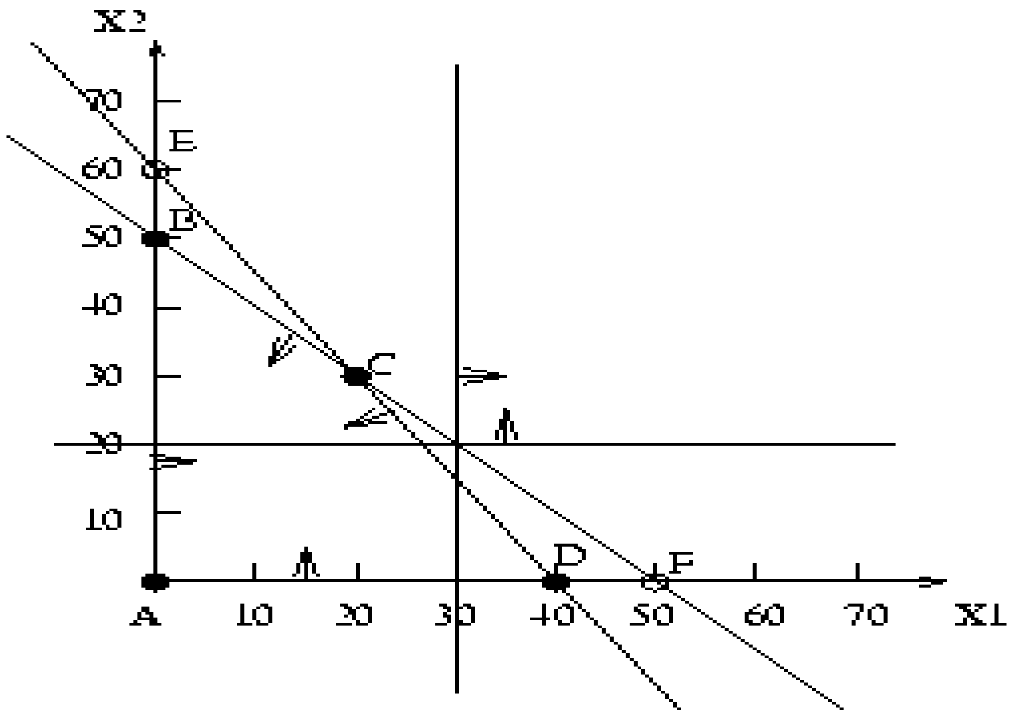


Figure 2: An infeasible LP

c) Unbounded 2-var LP's

Example 3.

Fresh Products Ltd. Is engaged in the business of breading cows quits farm. Since it is necessary to ensure a particular level of nutrients in their diet, Fresh Product Ltd. Buys two products P1 & P2 the details of nutrient constituents in each of which are as follows:

Nutrient Type	Nutrient Constituents in the product		Minimum nutrient requirements
	P_1	P_2	
A	36	6	108
B	3	12	36
C	20	10	100

The cost prices of both P1 & p2 are Rs. 20 per unit & Rs. 40 per unit respectively.

Formulate as a linear programming model & solve graphically to ascertain how much of the products P1 & p2 must be purchased so as to provide the cows nutrients not less then the minimum required?

Solution

Step 1

Mathematical formulation of the problem

Let x1 and x2 be the number of units of product P1 and p2.

The objective is to determine the value of these decision variables which yields the minimum of total cost subject to constraints. The data of the given problem can be summarized as bellow:

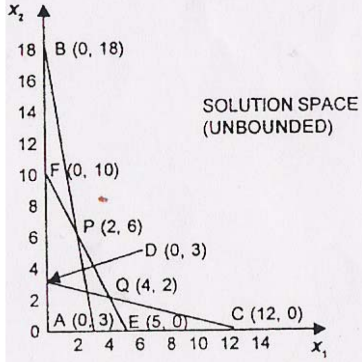
Decision Variables	*Number of Product*	*Type of Nutrient Constituent*			*Cost of Product*
		A	B	C	
X_1 X_2	1 2	36 6	3 12	20 10	20 40
Min. Requirement		108	36	100	

The above problem can be formulated and follows:
Minimize $Z = 20x_1 + 40x_2$ subject to the constraints:

$$36x_1 + 6x_2 \geq 108; 3x_1 + 12x_2 \geq 36; 20x_1 + 10x_2 \geq 0, x_2 0$$

Step 2

Graph the Constraints Inequalities. Next we construct the graph by drawing a horizontal and vertical axes, viz., x_1 axes in the Cartesian X1 OX2 plane. Since any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only, search for the desired pair (x_1, x_2) is restricted to the points of the first quadrant only.



The constraints of the given problem are plotted as described earlier by treating them as equations:

$$36x_1 + 6x_2 = 108; 3x_1 + 12x_2 = 36; \textit{and } 20x_1 + 10x_2 = 100$$

Since each of them happened to be 'greater than or equal to type', the points (x1,x2) satisfying them all will lie in the region that falls towards the right of each of these straight lines.

The solution space is the intersection of all these regions in the first quadrant. This is shown shaded in the adjoining figure.

Step 3

Locate the solution point. The solution space is open with B,P,Q and C as lower points.
Note 1. For ≥ constraints soln. Space is above the constraint line, as space
2. Now, consider only corner Pb of the sdn. Space according to LP therein.

Step 4

Value of objective function at corner points.

Corner points	Co-ordinates of Corner Points (x1,x2)	Objective function $Z = 20x_1 + 40x_2$	Value
B	(0,18)	20(0) + 40(18)	720
P	(2,6)	20(2) + 40(6)	280
Q	(4,2)	20(4) + 40(2)	160
C	(12,0)	20(12) + 40(0)	240

Step 5

Optimum value of the objective function . Here, we find that minimum cost of Rs. 160 is found at point Q(4,2), i.e., x1 = 4 and x2 = 2

Hence the first should purchase 4 units of product P1 and 2 units of product P2 in order to maintain a minimum cost of Rs. 160.

In the LP's considered above, the feasible region (if not empty) was a bounded area of the (X_1, X_2)-plane. For this kind of problems it is obvious that all values of the LP objective function (and therefore the optimal) are bounded. Consider however the following LP:

Example 4.

$$\max f(X_1, X_2) := 2X_1 - X_2$$

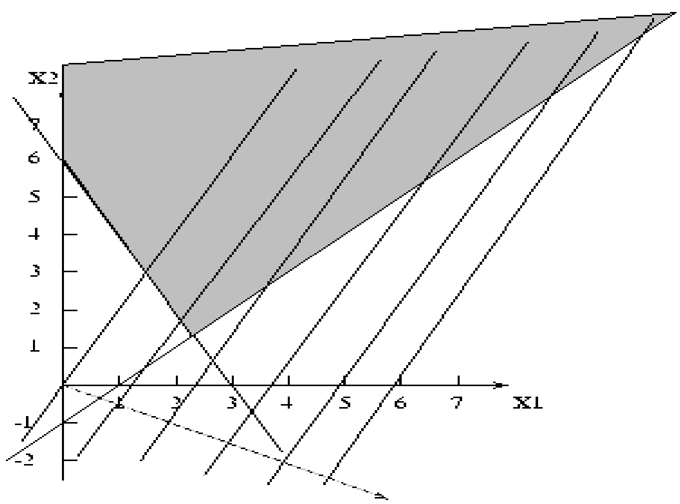
s.t.

$$X_1 - X_2 \leq 1$$

$$2X_1 + X_2 \geq 6$$

$$X_1,\ X_2 \geq 0$$

The feasible region and the direction of improvement for the isoprofit lines for this problem are given in Figure



An unbounded LP

It is easy to see that the feasible region of this problem is unbounded, and furthermore, the orientation of the isoprofit lines is such that no matter how far we ``slide'' these lines in the direction of increasing the objective function, they will always share some points with the feasible region. Therefore, this is an example of a (2-var) LP whose objective function can take arbitrarily large values. Such an LP is characterized as *unbounded*. Notice, however, that even though an unbounded feasible region is a necessary condition for an LP to be unbounded, it is not sufficient; to convince yourself, try to graphically identify the optimal solution for the above LP in the case that the objective function is changed to: $\max f(X_1, X_2) := -X_2$.

Summarizing the above discussion, I have shown that a 2-var LP can either

- ***have a* unique *optimal solution which corresponds to a ``corner" point of the feasible region, or***
- ***have* many *optimal solutions that correspond to an entire ``edge" of the feasible region, or***
- ***be* unbounded*, or be* infeasible*.***