# Unit 1

## Lesson 4: Graphical solution to a LPP

## Learning Outcomes

- **How to get an optimal solution to a linear programming model using Iso profit (or Iso cost method)**

## Iso profit or Iso cost method for solving LPP graphically

The term Iso-profit sign if is that any combination of points produces the same profit as any other combination on the same line. The various steps involved in this method are given below.

1. Identify the problem- the decision variables, the objective and the restrictions.

2. Set up the mathematical formulation of the problem.

3. Plot a graph representing all the constraints of the problem and identify the feasible region. The feasible region is the intersection of all the regions represented by the constraint of the problem and is restricted to the first quadrant only.

4. The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

5. Choose a convenient profit (or cost) and draw iso profit (iso cost)line so that it falls within the feasible region.

6. Move the iso profit (iso cost line to itself farther (closer) from (to) the origin.

7. Identify the optimum solution as the coordinates of that point on the feasible region touched by the highest possible iso profit line( or lower possible cost line).

8. Compute the optimum feasible solution.

**Let us do some examples to understand this method more clearly.**

**Example 1**

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximize the combined sum of the units of X and the units of Y in stock at the end of the week.

**Formulate the problem of deciding how much of each product to make in the current week as a linear program.**

**Solve this linear program graphically**.

**Solution**

Let

- x be the number of units of X produced in the current week
- y be the number of units of Y produced in the current week

then the constraints are:

- $50x + 24y \leq 40(60)$ machine A time
- $30x + 33y \leq 35(60)$ machine B time
- $x \geq 75 - 30$
- i.e. $x \geq 45$ so production of X $\geq$ demand (75) - initial stock (30), which ensures we meet demand
- $y \geq 95 - 90$
- i.e. $y \geq 5$ so production of Y $\geq$ demand (95) - initial stock (90), which ensures we meet demand

The objective is: maximize $(x+30-75) + (y+90-95) = (x+y-50)$
i.e. to maximize the number of units left in stock at the end of the week

It is plain from the diagram below that the maximum occurs at the intersection of
x=45 and 50x + 24y = 2400
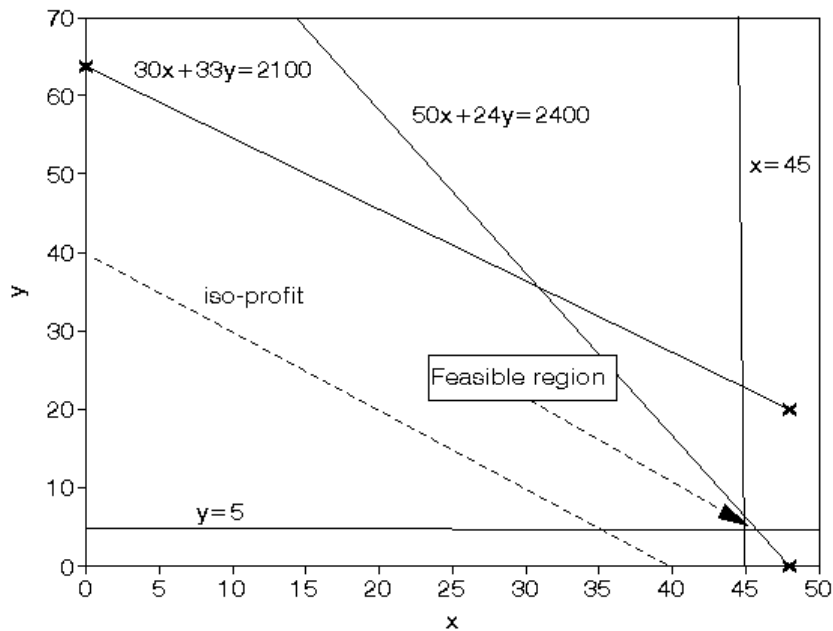


Solving simultaneously, rather than by reading values off the graph, we have that
x=45 and y=6.25 with the value of the objective function being 1.25

**Example 2.**
A firm manufactures & sells two product p1 & p2 at a profit of Rs. 45 per unit &
Rs. 80 per unit respectively. The quantities of raw materials required for both p1 & p2 are
given below:

| Raw Materials | Product | |
|---|---|---|
| | $P_1$ | $P_2$ |
| R1 | 5 | 20 |
| R2 | 10 | 15 |

**The maximum availability of both R1 & R2 is 400 & 450 units respectively.**
**You are required to formulate as a LP model & solve using the graphical method.**

**Solution**
To find the second point let us assume that the entire amount of input A is used to
produce the produce I and no unit of product II is produced, i.e., X2 = 0. Therefore $x_1 \leq$
i.e., the firm can produce either 80 or less then 80 units of product I. If we take the

maximum production x1 = 80. So the second point is (80,0) and this point Q in graph denotes product of 80 units of product I and zero unit of product II .
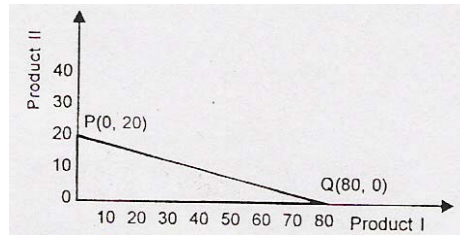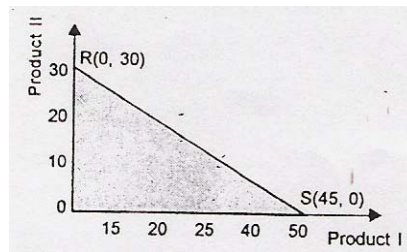


By joining the two points P (0,20) and Q (80,0) we get a straight line shows the maximum quantities of product I and product II that can be produced with the help of input A. The area POQ is the graphic representation of constraint $5x_1 20x_2 \leq 400$. It may be emphasized here that the constraint is represented by the area POQ and not be the line PQ. As far as the constraint of input A is concerned, production is possible at any point on the line PQ or left to it [ dotted area in the graph.]

In a similar way the second constraint $10x_1 15x_2 \leq 450$.

45, 0 be drawn graphically. For this purpose we obtain two points as follows:

(a) If production of product I is zero, the maximum production of product II is units. Point R ( 0,30) represents this combination in graph.



(b) If production of product II is zero, the maximum production of product I is 45 units. Point S ( 45, 0) in the graph represents this combination.

By joining points R and S we get a straight line RS. This line again represents the maximum quantities of product I and II that can be produced with the help of input B. ROS represents the feasibility region as far as the input B is concerned.

After plotting the two constraints next step is to find out the feasibility region. Feasibility region is that region in the graph which satisfies all the constraints simultaneously. Region POST in third graph represents feasible region. This region POST satisfies first constraint as well as second constraint. ROS is the feasible region

under first constant. But out of this RPT region does not fulfill second constraint. In a same way region POQ is the feasibility under constraint second but out of this, region TSQ does not satisfy constraint first.
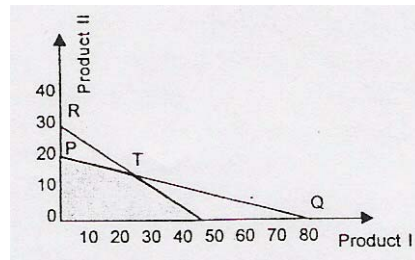


Region POST thus satisfies first constraint as well as second constraint and is thus feasible region. Each point in the feasible region POST satisfies both the linear constraints and is therefore a feasible solution. Non-negative constraints are also being satisfied in this region because we are taking first quadrant of the graph in which both the axes are positive. The feasible region is covered by the linked boundary PTS.

The corner points on the kinked boundary P, T and S are called as extreme points.

Extreme points occur either at the inter-section of the two constraints (T in this example' or at the intersection of one constraint and one axis ( P and S in our example ).

These extreme points are of great significance in optimal solution. Optimal solution will always be one of the these extreme points.

Final step in solving linear programming problem with the help of graph is to find out the optimal solution out of many feasible solution in the region POST. For this purpose we will have to introduce objective function into our graph. Our objective function is :-

$$Z = 45x_1 + 80x_2 \qquad \text{OR} \qquad 80x_2 = Z - 45x_1$$

$$x_3 = \frac{Z}{80} - \frac{45}{80}x_1 \qquad \text{OR} \qquad x_2 = \frac{Z}{80} - \frac{9}{16}x_1$$

Our objective function which is linear has negative slope.

It is plotted as dotted line $z_1z_1$ in the following $z_1z_1$ is in-fact Iso-profit line. The different combinations of I and product II of this line yield same profits ( say 20

units ) firm. Any line ( parallel to this line ) which is higher ( right to this $z_1z_1$ line signifies higher profit level ( say 25 units ) a line which is lower to the line $z_1z_1$ implies lower profit above graph; $z_1z_1$ line representing a specified level of profit be moved rightward still remaining in the feasible region. In words profits can be increase still remaining in the feasible.
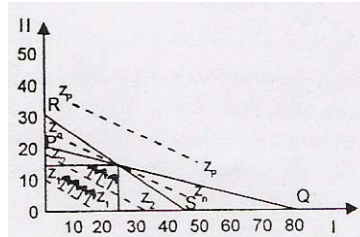


However, this is possible only up to $z_nz_n$. Different combination on $z_nz_n$ give the first a same level of profits. Point T (24,14) is in the feasible region POST and also on line $z_nz_n$. In other words this point 'T' represents a combination of the two products which can be produced under the combination of inputs and profits are maximum possible. Point T represents optimum combination. $z_nz_n$ iso-profit line, though signify higher profit is not in the feasible region. The firm, therefore, should produce 24 units of product I and 14 units of product II. Its profits will be maximum possible, equal to :-

$$Z=45x_1+80x_2 = (45)(24) +(80)(14)$$

$$= 1080 +1120 =Rs. \ 2200.$$

Optimum combination can be found also, without the introduction of profits function in our graph. As written above the corner points. P, T and S of the kinked cover of feasible region POST are called as extreme point and optimum T and S yields Maximum profit to the firm.

Point P, $x_1 = 0 \ x_2 = 20$

$$Z=45x_1+80x_2 = 45(0) +80(20) = \text{Rs. } 1600$$

Point T, $x_1 = 24$, $x_2 = 0$

$$Z=45x_1+80x_2 = (45)(24) +(80)(14) = \text{Rs. } 2200$$

Point S, $x_1=45$, $x_2 = 0$

$$Z=45x_1+80x_2 = (45)(245 +(80)(0) = \text{Rs. } 2025$$

**So combination T (24,14) is the intimation and Rs. 2200 is the maximum possible level of profits.**

**Example 3.**

**A firm produces two components which are then assembled into a final product. The cost per unit of these components is 0.60 and 1.00 respectively. The minimum amount of the various grades of raw material require3d for the manufacturing process are as follows:**

| Components | Raw material required | | |
|---|---|---|---|
| | $R_1$ | $R_2$ | $R_3$ |
| $C_1$ | 10 | 5 | 2 |
| $C_2$ | 4 | 5 | 6 |

Due to the requirement for a better quality product, the minimum usage value of the raw materials should be 20 units, 20 units & 12 units respectively.

Formulate as a LP problem & solve graphically.

**Solution**

The objective function is to minimize the total cost of production of both the components $C_1$ & $C_2$

Minimize $Z=0. 60x_1+1.00x_2$

Subject to:

$$10x_1 + 4x_2 \geq 20$$
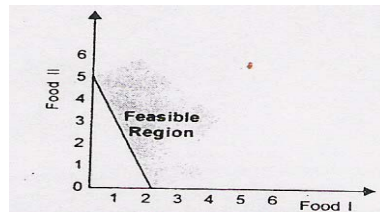
$$5x_1 + 5x_2 \geq 20$$
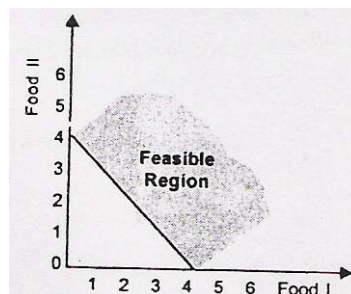
$$2x_1 + 6x_2 \geq 12$$

$$x_1, x_2 \geq 20$$

First constraint, i.e., constraint of nutrient A

$$10x_1 + 4x_2 \geq 20$$

If we consume no unit of food , I i.e., xi=0 then for satisfying the first constraint we will have to take either, 5 units or more than 5 units of good II. If



We consume minimum units of good II for minimizing cost ) then x2 = 5, In this way we get the first point x1(0,5) in first graph. Similarly if we do not take

Food II, i.e., x2=0, the minimum requirement of food 1 is 2 for satisfying the
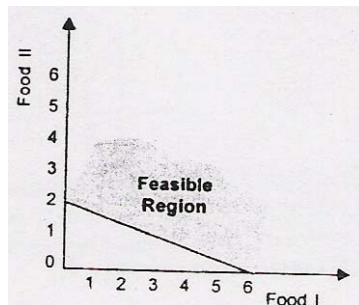


First constraint. In this way we get the second point Q (2,0) in the first graph. The line PQ represents the minimum quantities of food I and II that must be taken to satisfy the first constraint. Thus dotted area in the first graph represents feasibility region as far as the first constraint is concerned.

   In the same way second and third constraints of nutrient B and C respectively are plotted in the second graph and the third graph. In a fourth graph the three constraints are plotted collectively. Shaded area in this graph represents that area where all the constraints are satisfied simultaneously.
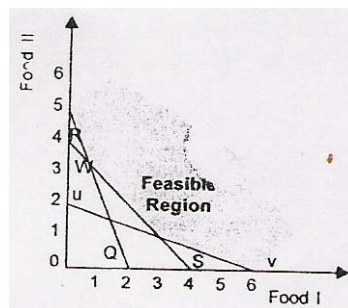


In other words, shaded area in the fourth graph is our feasible region. Feasible region is covered by kinked curve with P (0,5), W (0.66, .33) , T (3,1) and V (16, 0) as the extreme

points. Optimum combination is one of these extreme points. To find the optimum combination, i.e., a combination of good I and II which minimize cost as well as satisfy the three constraints, let us introduce the objective function into our graph. Our objective function is

$$T.C = 0.6x_1 + 1x_2$$

$$X_2 = T.C. - 0.6x_1 \text{ or } x_2 = T.C. - \frac{1}{2}x_1$$
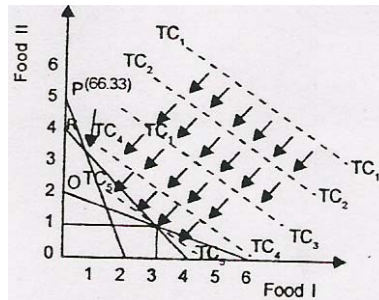


In our objectives function is a linear function with negatives slope equal to 1/2TC. TC in above graph represents the objective function. In fact $Tc_1$ $Tc_2$ is an iso-cost line a line representing the different combinations of food I and II having the same cost any of Rs. 20.

Any time which is higher to line Tci Tci implies higher cost and any line lower to the line Tci Tci means lower cost. The iso-cost line Tci Tci can be moved leftward still remaining in the feasible region. This means that cost can be reduced still remaining the feasible region.

However, this is possible up to Tcn Tcn. If iso-cost line is still moved leftward, it will go out of feasible region. All the combinations on line Tcn Tcn result same cost (which is minimum) but out of these combinations any combination T is in the feasible region. S.T (3.1) is the optimum combination. When three units of

food 1 and one unit of food II are taken all the constraints are satisfied and the cost is minimum possible. Minimum cost is.

T.C = 0.6xi + 1xi = (0.6) (3) + (1) (1) = Rs. 2.8

As in the case of maximisation in minimisation, we also can find optimum combination without introducing objective function in to our graph. As optimum point is one of the extreme points, use can use simple arithmetic to find which of these extreme points will result in minimum cost.

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Point  P  xi   =  O, x2 = 5
           T.C   =  0.6x1 + 1x2 (0.6) (0) + (1)5  = Rs. 5
Point  W  x1   =  0.66, x2  =  3.3
           T.C   =  0.6x1 + 1x2  =  (0.6) (0.66) + (3.3)
                =  Rs. 3.696
Point  T  x1   =  3, x2  =  1
            T.C.  =  0.6x1 + 1x2  =  (0.6) (3)  + (1) (1)
                =  Rs. 2.8
Point V  x1   =  6,  x2  = 0
           T.C.  =  0.6x1 + 1x2 =  (0.6) (6) + (1) (0) = Rs. 3.6
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Cost is minimum when x1 = 3  and x2 = 1.
Minimum cost is Rs. 2.8
(Students should note that answer is same)

# Case study

**Example 4**

A company is involved in the production of two items (X and Y). The resources need to produce X and Y are two fold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

|        | Machine time | Craftsman time |
|--------|--------------|----------------|
| Item X | 13           | 20             |
| Y      | 19           | 29             |

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at £10 per hour worked and craftsman time is costed at £2 per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is £20 for X and £30 for Y. The company has a specific contract to produce 10 items of X per week for a particular customer.

**Formulate the problem of deciding how much to produce per week as a linear program. Solve this linear program graphically**.

**Example 4**

A company manufactures two products (A and B) and the profit per unit sold is £3 and £5 respectively. Each product has to be assembled on a particular machine, each unit of product A taking 12 minutes of assembly time and each unit of product B 25 minutes of assembly time. The company estimates that the machine used for assembly has an effective working week of only 30 hours (due to maintenance/breakdown).

Technological constraints mean that for every five units of product A produced at least two units of product B must be produced.

**Formulate the problem of how much of each product to produce as a linear program. Solve this linear program graphically.**

**The company has been offered the chance to hire an extra machine, thereby doubling the effective assembly time available. What is the *maximum* amount you would be prepared to pay (per week) for the hire of this machine and why?**

**Example 5.**

**Solve the following linear program graphically:**

Maximize $5x_1 + 6x_2$

subject to

$x_1 + x_2 <= 10$

$x_1 - x_2 >= 3$

$5x_1 + 4x_2 <= 35$

$x_1, x_2 >= 0$

**Example 6.**

**A carpenter makes tables and chairs. Each table can be sold for a profit of £30 and each chair for a profit of £10. The carpenter can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Customer demand requires that he makes at least three times as**

**many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week.**

**Formulate this problem as a linear programming problem and solve it graphically.**

**Example 7.**

A cement manufacturer produces two types of cement, namely granules and powder. He cannot make more than 1600 bags a day due to a shortage of vehicles to transport the cement out of the plant. A sales contract means that he must produce at least 500 bags of powdered cement per day. He is further restricted by a shortage of time - the granulated cement requires twice as much time to make as the powdered cement. A bag of powdered cement requires 0.24 minutes to make and the plant operates an 8 hour day. His profit is £4 per bag for granulated cement and £3 per bag for the powdered cement.

**Formulate the problem of deciding how much he should produce as a linear program.**

**Solve this linear program graphically**.

*Today you have learnt about arriving at an optimal solution from a basic feasible solution. There may be an L.P.P. for which no solution exist or for which the only solution obtained is an unbounded one they are the special cases . In the next lecture I will discuss about these special cases.*