Unit 1

Lesson 3: Graphical method for solving LPP.

Learning outcome

1.Finding the graphical solution to the linear programming model

Graphical Method of solving Linear Programming Problems

Introduction

Dear students, during the preceding lectures, we have learnt how to formulate a given problem as a Linear Programming model.

The next step, after the formulation, is to devise effective methods to solve the model and ascertain the optimal solution.

Dear friends, we start with the graphical method and once having mastered the same, would subsequently move on to simplex algorithm for solving the Linear Programming model.
But let's not get carried away.

First thing first.
Here we go.

We seek to understand the IMPORTANCE OF GRAPHICAL METHOD OF SOLUTION IN LINEAR PROGRAMMING and seek to find out as to how the graphical method of solution be used to generate optimal solution to a Linear Programming problem.

Once the Linear programming model has been formulated on the basis of the given objective & the associated constraint functions, the next step is to solve the problem & obtain the best possible or the optimal solution various mathematical & analytical techniques can be employed for solving the Linear-programming model.

The graphic solution procedure is one of the method of solving two variable Linear programming problems. It consists of the following steps:-

Step I

Defining the problem. Formulate the problem mathematically. Express it in terms of several mathematical constraints & an objective function. The objective function relates to the optimization aspect is, maximisation or minimisation Criterion.

Step II

Plot the constraints Graphically. Each inequality in the constraint equation has to be treated as an equation. An arbitrary value is assigned to one variable & the value of the other variable is obtained by solving the equation. In the similar manner, a different arbitrary value is again assigned to the variable & the corresponding value of other variable is easily obtained.

These 2 sets of values are now plotted on a graph and connected by a straight line. The same procedure has to be repeated for all the constraints. Hence, the total straight lines would be equal to the total no of equations, each straight line representing one constraint equation.

Step III

Locate the solution space. Solution space or the feasible region is the graphical area which satisfies all the constraints at the same time. Such a solution point *(x, y)* always occurs at the comer. points of the feasible Region the feasible region is determined as follows:

(a) For" greater than" & " greater than or equal to" constraints (i.e.;), the feasible region or the solution space is the area that lies above the constraint lines.

(b) For" Less Then" &" Less than or equal to" constraint (ie;). The feasible region or the solution space is the area that lies below the constraint lines.

Step IV

Selecting the graphic technique. Select the appropriate graphic technique to be used for generating the solution. Two techniques viz; Corner Point Method and Iso-profit (or Iso-cost) method may be used, however, it is easier to generate solution by using the corner point method.

(a) **Corner Point Method**.

(i) Since the solution point *(x. y)* always occurs at the corner point of the feasible or solution space, identify each of the extreme points or corner points of the feasible region by the method of simultaneous equations.

(ii) By putting the value of the corner point's co-ordinates [e.g. (2,3)] into the objective function, calculate the profit (or the cost) at each of the corner points.

(iii) In a maximisation problem, the optimal solution occurs at that corner point which gives the highest profit.

In a minimisation problem, the optimal solution occurs at that corner point which gives the lowest profit.

Dear students, let us now turn our attention to the important theorems which are used in solving a linear programming problem. Also allow me to explain the important terms used in Linear programming.

Here we go.

IMPORTANT THEOREMS

While obtaining the optimum feasible solution to the linear programming problem, the statement of the following four important theorems is used:-

Theorems I.
The feasible solution space constitutes a convex set.

Theorems II.
within the feasible solution space, feasible solution correspond to the extreme (or Corner) points of the feasible solution space.

Theorem III.
There are a finite number of basic feasible solution with the feasible solution space.

Theorem IV
The optimum feasible solution, if it exists. will occur at one, or more, of the extreme points that are basic feasible solutions.

Note. Convex set is a polygon "Convex" implies that if any two points of the polygon are selected arbitrarily then straight line segment Joining these two points lies completely within the polygon. The extreme points of the convex set are the basic solution to the linear programming problem.

IMPORTANT TERMS

Some of the important terms commonly used is linear programming are disclosed as follows:

(i) **Solution**

Values of the decision variable *x;(i* = 1,2,3, in) satisfying the constraints of a general linear programming model is known as the solution to that linear programming model.

(ii) **Feasible solution**

Out of the total available solution a solution that also satisfies the non-negativity restrictions of the linear programming problem is called a feasible solution.

(iii) **Basic solution**

For a set of simultaneous equations in Q unknowns (p Q) a solution obtained by setting (P - Q) of the variables equal to zero & solving the remaining P equation in P unknowns is known as a basic solution.

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The variables which take zero values at any solution are detained as non-basic variables & remaining are known as-basic variables, often called basic.

(iv) **Basic feasible solution**

A feasible solution to a general linear programming problem which is also basic solution is called a basic feasible solution.

(v) **Optimal feasible solution**

Any basic feasible solution which optimizes (ie; maximise or minimises) the objective function of a linear programming modes known as the optimal feasible solution to that linear programming model.

(vi) **Degenerate Solution**

A basic solution to the system of equations is termed as degenerate if one or more of the basic variables become equal to zero.

I hope the concepts that we have so far discussed have been fully understood by all of you.

Friends, it is now the time to supplement our understanding with the help of examples.

Example 1.

X Ltd wishes to purchase a maximum of3600 units of a product two types of product a. & are available in the market Product a occupies a space of 3 cubic Jeet & cost Rs. 9 whereas occupies a space of 1 cubic feet & cost Rs. 13 per unit. The budgetary constraints of the company do not allow to spend more than Rs. 39,000. The total availability of space in the company's godown is 6000 cubic feet. Profit margin of both the product a & is Rs. 3 & Rs. 4 respectively. Formulate as a linear programming model and solve using graphical method. You are required to ascertain the best possible combination of purchase of a & so that the total profits are maximized.

Solution

Let x_1 = no of units of product α &
x_2 = no of units of product β

Then the problem can be formulated as a P model as follows:-

Objective function, Maximise $Z = 3x_1 + 4x_2$

Constraint equations: -

$$x_1 + x_2 \leq 3600 \ (Maximum\ Units\ Constraint)$$

$$3x_1 + x_2 \leq 6000$$ (Storage area constraint)

$$9x_1 + 13x_2 \leq 39000$$ (Budgetary constraint)

$$x_1 + x_2 \leq 0$$

Step I

Treating all the constraints as equality, the first constraint is

$$x_1 + x_2 = 3600$$

Put $x_1 = o \Rightarrow x_2 = 3600 \therefore$ The point is (0, 3600)

Put $x_2 = o \Rightarrow x_1 = 3600 \therefore$ The point is (3600, 0)

Draw is graph with x_1 on x-axis & x_2 on y-axis as shown in the figure.

Step II

Determine the set of the points which satisfy the constraint:

$x_1 + x_2 = 3600$

This can easily be done by verifying whether the origin(0,0) satisfies the constraint. Here,

$0 + 0 = 3600$. Hence all the points below the line will satisfy the constraint.

Step III

The 2nd constraint is: $3x_1 + x_2 \leq 6000$

Put $x_1 = o \Rightarrow x_2 = 6000$ and the point is (0, 6000)

Put $x_2 = o \Rightarrow x_1 = 200$ and the point is (200, 0)

Now draw its graph.

Step IV

Like it's in the above step II, determine the set of points which satisfy the constraint $3x_1 + x_2 \leq 6000$. At origin;

$0 + 0 < 6000$. Hence, all the points below the line will satisfy the constraint.

Step V

The 3rd constraint is: $9x_1 + 12x_2 \leq 39000$

Put $x_1 = o \Rightarrow x_2 = 3000$ & the point is (0, 3000)

Put $x_2 = o \Rightarrow x_1 = \frac{13000}{3} \& the\ po\operatorname{int} is \left(\frac{13000}{3}, 0 \right)$

Now draw its graph.

Step VI

Again the point (0,0) ie; the origin satisfies the constraint $9x_1 + 12x_2 \leq 39000$.

Hence, all the points below the line will satisfy the constraint.

Step VII

The intersection of the above graphic denotes the feasible region for the given problem.

Step VIII

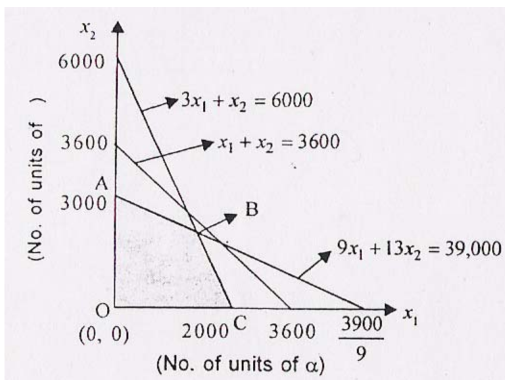
Finding Optimal Solution

Always keep in mind two things: -

(i) For $\geq$ constraint the feasible region will be the area, which lies above the constraint lines, and for $\leq$ constraints, it will lie below the constraint lines.

This would be useful in identifying the feasible region.

(ii) According to a theorem on linear programming, an optimal solution to a problem (if it exists) is found at a corner point of the solution space.



Step IX

At corner points (O, A, B, C), find the profit value from the objective function. That point which maximize the profit is the optimal point.

Corner Point	W-Ordinates	***Objective Func*** $Z = 3x_1 + 4x_2$	Value
O	(0,0)	Z=0+0	0
A	(0,3000)	Z=0+4x3000	12,000
C	(2000,0)	Z=0+3x2000+0	6,000

For point B, solve the equation $9x_1 + 12x_2 \leq 39000$

And $3x_1 + 6x_2 \leq 6000$ to find point B

($\therefore A + B$, these two lines are intersecting)

ie, $3x_1 + x_2 = 6000$ …(1)

$9x_1 + x_2 = 39000$ …(2)

Multiply equ (i) by 3 on both sides:

$\Rightarrow 9x_1 + 3x_2 = 18000$ …(3)

$9x_1 + 13x_2 = 39000$ …(4)

$- \quad - \quad -$

$-10x_2 = -21000 \qquad \therefore x_2 = 2100$

Put the Value of x_2 in first equation:

$\Rightarrow x_1 = 1300$

At point (1300,2100)

$Z = 3x_1 + 4x_2$

$= 3x1300 + 4x2100$

= 12,300 which is the maximum value.

Result

The optimal solution is:

No of units of product $\alpha = 1300$

No of units of product $\beta = 2100$

Total profit, = 12300 which is the maximum.

Well friends, it's really very simple.

Isn't it?

Let's consider some more examples.

Example 2.

Greatwell Ltd. Produces & sell two different types of products P1 & P2 at a profit margin of Rs. 4 & Rs. 3 respectively. The availability of raw materials the maximum no of production hours available and the limiting factor of P2 can be expressed in terms of the following in equations:

$$4x1 + 2x2 \leq 10$$

$$2x1 + \frac{8}{3}x2 \leq 8$$

$$x2 \leq 6$$

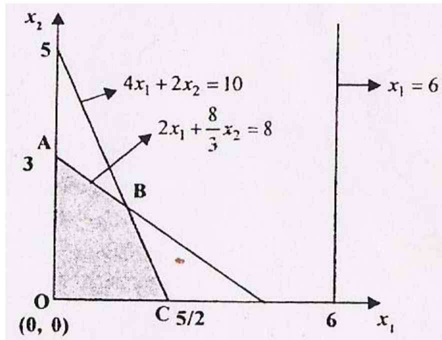
$$x1, x2 \geq 0$$

Formulate & solve the LP problem by using graphical method so as to optimize both P_1 & P_2.

Solution

Objective: Maximise $Z = 4x_1 + 3x_2$

Since the origin (0,0) satisfies each and every constraint, all points below the line will satisfy the corresponding constraints.



Consider constraints as equations & plot then as under:

$4x1 + 2x2 \leq 10$

put $x1 = 0 \Rightarrow x = 5$ and the point is (0,5) …(1)

Put $x2 = 0 \Rightarrow x = 5/2$ and the point is (5/2, 0)

$2x1 + \frac{8}{3}x2 \leq 8$ …(2)

put $x1 = 0 \Rightarrow x2 = 3$ and the point is (, 3)

put $x2 = 0 \Rightarrow x1 = 4$ and the point is (4, 0)

$x_1 = 6$ …(3)

The area under the curve OABC is the solution space. The constraint x_1=6 is not considered since it does not contain the variable x_2.
Getting optimal solution.

Corner Point	Co-ordinates	Obj.-Func. $Z = 4x_1 + 3x_2$	Value
0	(0,0)	Z=0+0	0
A	(0,3)	Z=0+3x3	9
B	*	*	*
C	(5/2,0)	$Z = 4x\frac{5}{2}x_2 = 8$	10

Point B is the intersection of the curves $4x_1 + 2x_2 = 10$ &

$$2x_1 \frac{8}{3}x_2 = 8$$

Solving as system of simultaneous equation:
Point B is (8/5, 9/5

$\therefore Z = 4x_1 + 3x_2$

$= 4(8/5) + 3(9/5) = 59/5$

Result
The optimal solution is:
Not of units of P2 = 9/5
Total profit = 59/5
Remarks. Since 8/5 & 9/5 units of a product can not be produced , hence these values must be rounded off. This is the limitation of the linear programming technique.

Dear students, we have now reached the end of our discussion scheduled for today.
See you all in the next lecture.
Bye.

Dear students, I hope all of us have by now properly grasped the intricacies involved in the graphical method.

It's now time to supplement our understanding of the concept by taking various examples.

Here we go.

Example 3.

Due to the unavailability of desired quality of raw materials, ABC Ltd. Can manufacture maximum 80 units of product A & 60 units of product B. A consumes 5 units and B consumes 6 units of raw materials in the manufacture respectively and their respective profit margins one Rs. 50 & Rs. 80.

Further, A requires 1 man-day of labour per unit & B requires 2 man-day of labour per unit. The constraints operating are:

Supply of raw material - maximum 600 units
Supply of labour - maximum 160 man-days.
Formulate as a LP model & solve graphically.

Solution

The question can be presenter in a tabular form as under:

Product	Raw Material Required	Labour Required	Maximum Units Produced	Profit Per Unit
A	5	1	80	50
B	6	2	60	80

Let x1 = number of units of Product A
x2 = number of units of Product B

Since the objective of the company is to maximize its profit. Then the model can be stated as follows:

Maximize Z = 50x1 + 80x2

Subject to the linear constraints

x1 ≤ 80
Functional restrictions
x2 ≤ 60

5x1 + 6x2 ≤ 600 (Electronic restrictions)

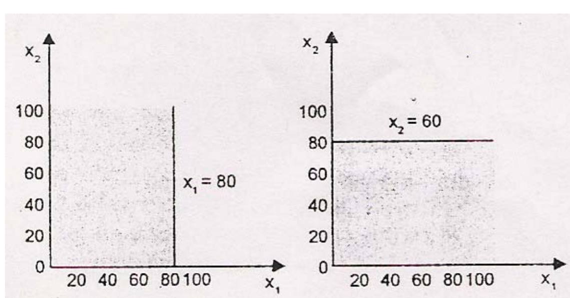
x1 , 2x2 ≤ 160 (Labour supply)
and x1, x2 ≥ 0

Step 2

Graph the constraint inequalities

For the problem, x1 axis will represent product. A and B respectively. Each inequality will be treated as an equality and then their respective intercepts on both the axis can be determined.

For the first two constraints, we have x1= 80 and x2 = 60, i.e., draw a graph with production of product A and product B, as shown below in Fig. 2-1 and Fig 2-2.

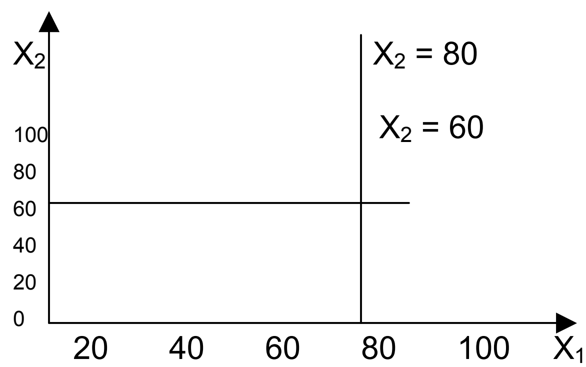


Since no more than 80 units of x1 and 60 units of x2 can be produced per day i.e., x1≤ 80 and x2≤60, it

follows that
$x1 \le 80$ or $x1 < 80$ and $x2 \le 60$ $x2 < 60$

Thus any point on the line (the equality sign) and other points within the shaded area (the less than sign) will satisfy as shown in Fig. 2-3

Now consider the inequality $5x1 + 6x2 \le 600$. Treating it as a equality we have

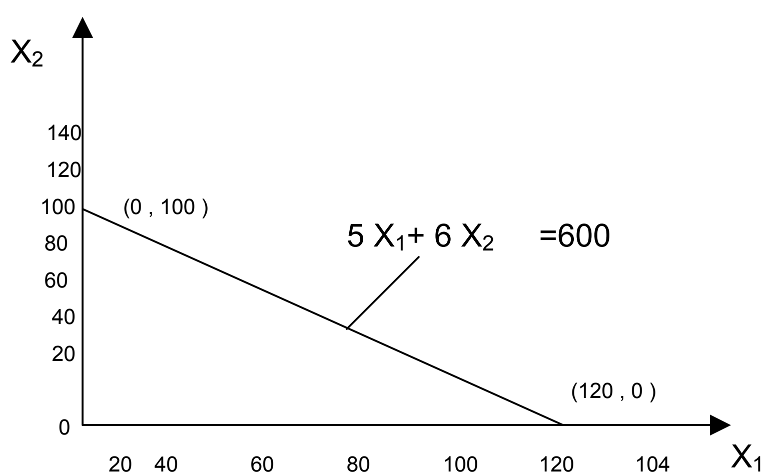


$$5\ X_1 + 6\ X_2 = 600$$

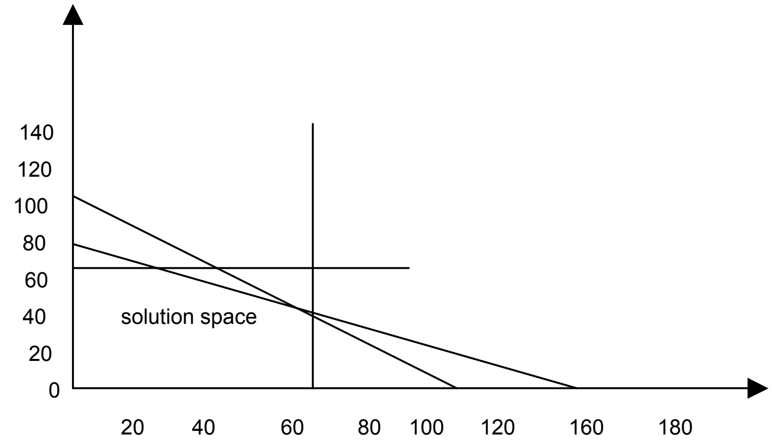
or $$X_1 /(600)/5 \ + X_2\ (600)\ /\ 6 \qquad = 1$$

or $$X_1/200 + X_2\ /\ 100 = 1$$

draw a straight line by joining 120 on X_1 axis and 100 on X_2 axis (as shown in fig 2-4) thus the set of point s satisfying $X_1 \ge 0$, X_2 # 0 and the constraint is represent by the shaded area as shown below:



similarly we can draw a graph for $X_1 + 2 X_2 \leq 160$



when all the constraint are imposed together, the value of X_1 and X_2 can lie only in the shaded area. The area which is bounded by all the constraints lines including all the boundary points is called the feasible region or solution space. This is shown in fig 2-5 by the shaded area OABCDE.

Step 3
Locate the solution point
Since the value of X_1 and X_2 have to lie in the shaded area which contains an infinite number of points would satisfy the constrains of the virgin L>P>P but we are confined only to those points which corresponds to corners of solution space. Thus as shown in fig. 2-5 the corner points of feasible region are 0 = (0,0) A = (80,0), B = (80,33,33), C = (60,50), D = (40,60)and E = (0,60)

Step 4

Value of objective function at corner points

Corner points	Co-ordinates of corners points (X_1, X_2)	Objective $Z = 50 X_1 + 80 X_2$	Value
O	(0,0)	0+0	0
A	(80,0)	50(80)+0(0)	4000
B	(80,33.33)	50(80)+ (80,33.33)	6666.40
C	(60,50)	50(60) +80(50)	7000
D	(40,60)	50(40) +80(60)	6800
E	(0,60)	50(0) +80(60)	4800

Step 5
Optimal value of the objective function.

Here we see that the maximum profit of Rs. 7000 is obtained at the point C = (60,50) I,e, $X_1 = 60$ and $X_2 = 50$ which together satisfy all the constraints.

Hence to maximize profit, the company must produce 60 units of product A and 50 units of product B.

Example4.

Alpha ltd. produces two products X and Y each requiring same production capacity. The total installed production capacity is 9 tones per days. Alpha Ltd. Is a supplier of Beta Ltd. Which must supply at least 2 tons of X & 3 tons of Y to Beta Ltd. Every day. The production time for X and Y is 20 machine hour pr units & 50 machine hour per unit respectively the daily maximum possible machine hours is 360 profit margin for X & Y is Rs. 80 per ton and Rs. 120 per ton respectively. Formulate as a LP model and use the graphical method of generating the optimal solution for determining the maximum number of units of X & Y, which can be produced by Alpha Limited.

Solution

Objective function

Maximize (total profit) $Z = 80 X_1 + 120 X_2$

Subject to the constraints:

$X_1 + X_2 \leq 9$, $X_1 \geq 2$, $X_2 \geq 3$ (Supply constraint)

$20 X_1 + 50 X_2 \leq 360$ (Machine hours constraint)

and

$X_1 \geq$, $X_2 \geq 0$

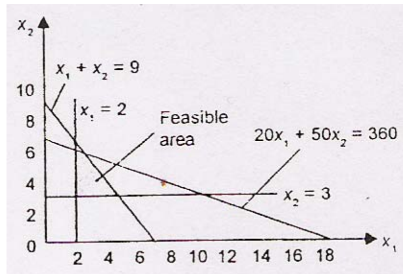
Where;

x_1 = Number of units (in tones) of Product X.

x_2 = Number of units (in tones) of Product Y.

Now the region of feasible solution shown in the following figure is bounded by the graphs of the Linear equalities:

$x_1 + x_2 = 9, x_1 = 2, x_2 = 3 \, and \, 20x_1 + 50x_2 = 360$ and by the coordinates axes.



The corner points of the solution space are:

A(2,6.4), B (3,3) and D(2,3)

The value of the objective function at these corner points can be determined as follows:

Corner Points	Co-ordinates of Corner Points (x_1, x_2)	Objective Function $Z=80x_1+20x_2$	Value
A	(2,6.4)	(80(2) +120(6.4))	928
B	(3,6)	(80(3) +120 (6))	960
C	(6,3)	(80(6) +120(3))	840
D	(2,3)	(80(2) +120(3))	520

The maximum profit (value of Z) of Rs. 960 is found at corner point B i.e., x1=3 and x2 = 6. Hence the company should produce 3 tonnes of product X and 6 tonnes of product Y in order to achieve a maximum profit of Rs. 960.

Example 5.

Unique Car Ltd. Manufacturers & sells three different types of Cars A, B, & C. These Cars are manufactured at two different plants of the company having different manufacturing capacities. The following details pertaining to the manufacturing process are provided:

Manufacturing Plants	Maximum Production (of Cars)			Operating cost of Plants
	A	B	C	
1	50	100	100	2500
2	60	60	200	3500
Demand (Cards)	2500	3000	7000	

Using the graphical method technique of linear programming , find the least number of days of operations per month so as to minimize the total cost of operations at the two plants.

Solution

Let x_1 = no-of days plant 1 operates; and

X_2 = no-of days plant 2 operates.

The objective of uniques Car Ltd. Is to minimize the operating costs of both its plants.

The above problem can be formulated as follows:

Ie; Minimize Z = $2,500x_1+3,500x_2$) Objective)

Subject to:

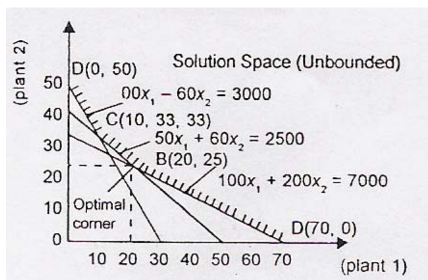
$$50x_1 + 60x_2 \geq 2{,}500$$

$$100x_1 + 60x_2 \geq 3{,}000$$

$$100x_1 + 200x_2 \geq 7{,}000$$

and $x_1, x_2 \geq 0$

Making the graphs of the above constraints:



The solution space lines at the points A,B,C & D Calculating the Optimal solution:

Points	***Co-ordinates***	***Objective – func***	***Value***
0	(0,0)	Z=0+0	0
A	(70,0)	Z=2500x70+0	1,75,000
B	(20,25)	Z=2500x20+3500x25	1,37,500
C	(10,33.33)	Z=2500x10+3500x33.33	1,41,655
D	(0,50)	Z=0+3500x50	1,75,000

Thus, the least monthly operately cost is at the point B. Where x1=20 days, x2=25 days & operating cost = Rs. 1,37,500.

Example 6.

The chemical composition of common (table) salt is sodium chloride (NACL). Free Flow Salts Pvt. Ltd. Must produce 200 kg of salt per day. The two ingredients have the following cost – profile:

Sodium (Na) - Rs. 3 per kg

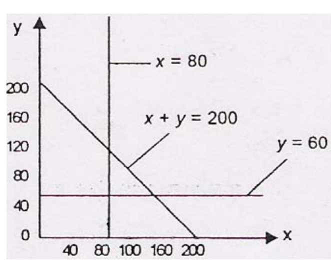
Chloride (CL)- Rs. 5 per kg

Using Linear programming find the minimum cost of salt assuming that not more than 80 kg of sodium and at lest 60 kg of chloride must be used in the production process.

Solution
Formulating as a LP model:

Minimise Z=3x+5y (objective)

Subject to:
$x+y = 200$ (Total 200 kg to be produced per day)
$x \leq 80$ (Sodium not to exceed 80 kg)
$y \geq 60$ (Chloride to be used at least 60 kg)
$x,y \geq 0$
Where
x = Qty of sodium required in the production &
y = Qty of chloride required in the production.
It is clear from the graph that there is no feasible solution area. It has only one feasible point having the co-ordinates (80,120)



Optimal solution:- x = 80, y = 120, and z=3x80+5x120=840 Thus, 80 kg of sodium & 120 kg of chloride shall be mixed in the production of salt at a minimum cost of Rs. 840.

Dear students, we have now reached the end of our discussion scheduled for today.
See you all in the next lecture.
Bye.